NON-LINEAR COEFFICIENT-WISE STABILITY AND HYPERBOLICITY PRESERVING TRANSFORMATIONS

A DISSERTATION SUBMITTED TO THE GRADUATE DIVISION OF THE UNIVERSITY OF HAWAI'I AT MĀNOA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

MAY 2012

By

Lukasz Grabarek

Dissertation Committee:

George Csordas, Chairperson
Thomas Craven
Erik Guentner
Marcelo Kobayashi
Wayne Smith

ACKNOWLEDGMENTS

I wish to thank, first and foremost, my thesis adviser Dr. George Csordas whose infinite patience, gentle tutelage, and limitless enthusiasm made this work possible. The subject matter considered in this dissertation as well as its presentation have markedly benefited from his scholarship, and if he is too modest to admit the former, certainly its effect on the style of my prose is evidence for the latter.

I wish to thank everyone who has ever alleviated the pecuniary woes of a young mathematician. In particular the Achievement Rewards for College Students (ARCS) foundation of Honolulu together with Mr. and Mrs. Robert and Doris Pulley for a most generous award of $5,000, the University of Hawai'i Graduate Student Organization for the travel reimbursement in the amount of $952, and the American Institute of Mathematics for hosting the 2011 workshop "Stability, hyperbolicity, and zero localization of functions" and generously funding travel and accommodation expenses. I am grateful for all the help I have received and promise that all these amounts will be repaid in kind as soon as I am able to do so.

I am grateful to my friends and colleagues at the mathematics department for their suggestions and comments regarding this dissertation. The members of the dissertation committee deserve a special thanks for taking the time out of their busy schedules to review and comment upon this work. Many errors and typos were culled as a result of the conversations I had regarding this work with Mr. Robert Bates, Dr. Matthew Chasse, Dr. Alex Gottlieb, and Mr. Rintaro Yoshida.

Finally, I wish to thank everyone who contributed indirectly to completion of this dissertation; you know who you are and you have my gratitude.

ABSTRACT

We study the operation of replacing the coefficients of a real function with a non-linear combination of its coefficients. We are particularly interested in the coefficient-wise transformations that preserve the location of zeros in a prescribed region. The earliest example of such transformations is given in the classical composition theorem of Malo, Schur, and Szegö, from which it follows that squaring all the coefficients of a real entire function with only real zeros preserves the reality of zeros. A next and quite recent example of such transformations is due to P. Brändén who studied the operation that subtracts from the square of a coefficient the product of its two closest neighbors. We relate this form of coefficient-wise transformation to a system of inequalities characterizing the Laguerre-Pólya class of real entire functions. Then using Brändén's criterion for stability preserving coefficient-wise transformations we describe a class of non-linear transformations that arise naturally from this system of inequalities. Other systems of inequalities characterizing the Laguerre-Pólya class would potentially yield coefficient-wise transformations of a different character. We investigate the stability preserving properties of coefficient-wise transformations that arise from the systems of inequalities characterizing the Laguerre-Pólya class.

Applications to logarithmic concavity and decreasing sequences yield novel results on the stability preserving properties of these transformations. Under certain conditions the stability-preserving properties of the coefficient-wise transformations under consideration extend to sequences of real stable polynomials. We highlight several open problems and propose conjectures regarding the stability-preserving properties of transformations considered in this work.

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

1.1 Special notation

We list here some of the special notation along with a brief description and the page number where it first occurs. Other special notation appears locally within statements of results, and because of its limited scope is not mentioned at this time.

$\mathbb{R}[x]$	The ring of real polynomials	4
$\mathbb{C}[x]$	The ring of complex polynomials	16
$\Re z$	The real part of z	6
$\Im z$	The imaginary part of z	10
$\mathscr{L}$-$\mathscr{P}$	Laguerre-Pólya class of real entire functions	5
$\mathscr{L}$-$\mathscr{P}^+$	Laguerre-Pólya class of real entire function with non-negative coefficients	5
$\mathscr{L}$-$\mathscr{P}_n^+$	Polynomials of degree at most n in $\mathscr{L}$-$\mathscr{P}^+$	15
$\mathscr{L}$-$\mathscr{P}^*$	Functions that are products of $p(x) \in \mathbb{C}[x]$ and $\varphi(x) \in \mathscr{L}$-$\mathscr{P}$	15
$\mathscr{L}$-$\mathscr{P}_n^*$	Polynomials of degree at most n in $\mathscr{L}$-$\mathscr{P}^*$	15
x_0	The positive zero of $x^3 - x^2 - 2x - 1$	56
$L_p(\varphi(x))$	Laguerre expression of length $2p$ associated to $\varphi(x)$	9
CZDS	Complex zero decreasing sequence	13
Λ_k	A non-linear transformation of a sequence $\{a_k\}_k^\infty$	2
Λ	A non-linear operator $a_k \mapsto \Lambda_k$	2
$\Lambda_k^p(\mu)$	Non-linear transformation of the sequence μ of order p	21
$\Lambda^p(\mu)$	The non-linear operator $a_k \mapsto \Lambda_k^p(\mu)$	21
T_k^p	Toeplitz-type transformation of order p	49
T_p	Non-linear operator $a_k \mapsto T_k^p$	49
${}_pF_q$	Hypergeometric function	25
$\left\{ {n \atop k} \right\}$	Stirling numbers of the second kind	67
$B_n(x)$	Bell polynomial of degree n	67
S_r	The non-linear operator $a_k \mapsto a_k^2 - a_{k-r}a_{k+r}$	30
$S(\lambda)$	The strip $\{z \in \mathbb{C} \,:\, \lvert\Im z\rvert < \lambda\}$	11

1.2 Synopsis

Let $f(z) = \sum_{k=0}^{\omega} a_k z^k, 0 \leq \omega \leq \infty$, be a real entire function. We consider the action of an operator $\Lambda : \mathbb{C} \to \mathbb{C}, a_k \mapsto \Lambda_k$, on the coefficients of $f(z)$. Here, and throughout the sequel, Λ_k denotes any expression that may be obtained by summing and multiplying together coefficients of $f(z)$ (see Notation 36). Suppose that the zeros of $f(z)$ lie in a prescribed region, what can be said about the non-linear operators Λ that act on the coefficients of $f(z)$ and preserve the location of its zeros?

Chapter 1 introduces the notation, terminology, and auxiliary results pertinent to the discussion of non-linear stability and hyperbolicity preserves. We give a brief history of the theory of stability and the development of the branch of this theory that leads to problems related to non-linear stability and hyperbolicity preservers.

In Chapter 2 we consider a form of a non-linear operator and extend P. Brändén's non-linear operator (cf. Theorem 35) to a family of such examples. (Remarkably, the hyperbolicity preserving properties of this class of non-linear operators are constructed from the classical systems of inequalities that characterize the class of real entire functions locally uniformly approximated by real polynomials with only real and non-positive zeros. The contents of Chapter 2 are to appear, in part, in [43] which will then exist in two versions. The proof of Theorem 42 given in [43] is the same as presented in this work, but a somewhat less elegant presentation exists in the ArXiv preprint archive. Chapter 3 deals with higher order analogues of non-linear operators considered in Chapter 2. We present an alternate characterization of the real sequences that generate the class of non-linear operator encountered in Chapter 2. Implications, applications, and examples of the present work are found in Chapter 4. We consider the concavity preserving properties of non-linear term-wise operations. We conclude with a list of open problems, conjectures, and directions for further research.

Some theorems will be stated without proof, in the absence of which we provide well-established references. The main results of this dissertation are found in:

Theorem 42

We exhibit a class of non-linear coefficient-wise transformations that extend P. Brändén's non-linear operator $a_k \mapsto a_k^2 - a_{k-1}a_{k+1}$.

Theorem 53

The class of non-linear coefficient-wise transformations given in Theorem 42 is further described in terms of real polynomials with all real non-positive zeros.

Theorem 59

The class of non-linear coefficient-wise transformations given in Theorem 42 is further described in terms of its generating polynomials. We show that the set of generating polynomials of this class of non-linear coefficient-wise transformations is closed under certain symmetry-preserving operations on the set of its zeros.

Theorems 81 and 87

Concavity properties of iterated transformations on decreasing sequences are shown to be multiplicative. A characterization of log-concave sequences is given in terms of concavity properties of its iterated sequence.

Theorem 92

The concavity preserving properties of the non-linear coefficient-wise transformation $a_k \mapsto a_k^2 - a_{k-1}a_{k+1}$ are extended to the operator $p_n \mapsto p_{n+1}^2 - p_n p_{n+2}$, where p_n is a real polynomial of degree n.

Lemma 103 and Corollaries 104 and 105

A lower bound is given on the Turán ratios of the coefficients of real polynomials with highly concave coefficient sequences and all zeros contained in strip.

Lemma 109

The Hadamard product of the non-linear operators $\Lambda^1(z^2+1)$ and T^3, acting on stable polynomials, is shown to be stable.

In addition, to the author's best knowledge the observations made in Propositions 31, 49, 48, 66, 106, 111, 112; Lemmas 46, 68, 108; Corrolaries 47, 55, 56, 63, 80, 82, 84, 88, 89, 93; and Theorem 33 appear to be new.

1.3 Historical remarks

In the theory of the location and distribution of zeros of real polynomials and real transcendental entire functions the following problem, suggested by E. Laguerre in 1884 (see for example [55, p. 382] or [64, p. 116]), begat a vast literature on the effect of transformations on entire functions that preserve the location of zeros.

Problem 1. *Characterize all real sequences* $\{\gamma_k\}_{k=0}^{\infty}$ *such that*

$$Z_c\left(\sum_{k=0}^{n}\gamma_k a_k x^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_k x^k\right), \tag{1.1}$$

where $Z_c(p(x))$ *denotes the number of non-real zeros of* $p(x)$*, counting multiplicities.* □

The problem received due consideration from eminent mathematicians and in their seminal 1914 paper ([75]), G. Pólya and J. Schur completely characterized the linear operators, acting diagonally on the monomial basis of $\mathbb{R}[x]$, which preserve the reality of zeros of real polynomials and called them *multiplier sequences* (see §1.4.1). This characterization of multiplier sequences in terms of a class of entire transcendental functions, the *Laguerre-Pólya* (see §1.4) class and a class of real polynomials, the *Jensen polynomials* (see §1.4.1), was subsequently hailed by R.P. Boas as a "key result on the boundary between Algebra and Analysis" [3, p. 418]. The significance of the Laguerre-Pólya class is canonical as it consists exactly of the real entire functions that are the uniform limits, on compact subsets of $\mathbb{C}$, of real polynomials all of whose zeros are real ([65, Ch. VIII], [71, p.10], [74, p. 54], [74, p. 105, Satz I]). Thus, necessary and sufficient conditions for membership in the Laguerre-Pólya class have been of special interest in analysis, especially due to the connection with the celebrated Riemann Hypothesis (see [23], [24], [25], [26], [29], [30], [58]). In spite of all of this, Problem 1 remains open as there is no known characterization of multiplier sequences that satisfy (1.1).

The Laguerre-Pólya class may be characterized by means of systems of inequalities, though few such characterizations are known (cf. §1.4). However, in certain settings the Laguerre-Pólya class occurs naturally and we focus here on two notions central to the sequel. First, a classical theorem of E. Laguerre (Theorem 21), asserting the existence of non-trivial sequences with the property (1.1), is the most natural motivation for the Laguerre-Pólya class. The linear operators described in statement (i) of Laguerre's Theorem are real sequences interpolated by real polynomials of degree at most n with all real zeros that lie outside the interval $[0, n]$. The closure of such polynomials is precisely the Laguerre-Pólya class $\mathscr{L}\text{-}\mathscr{P}(-\infty, 0]$ (see §1.4, [77]), and this is statement (iii) of Laguerre's Theorem. Indeed, if if $h(x)$ is a polynomial with all real and negative zeros, then the sequence $\{h(k)\}_{k=0}^{\infty}$ is *a multiplier sequence of the first kind* (see §1.4.1). Second, one may arrive at the condition that all zeros of a function are real by considering mappings of the upper to the lower half-plane and it is here that the notion of *stability* comes into view.

P. Brändén, in his proof of a conjecture of S. Fisk, R. P. Stanley, P. R. W. McNamara and B. E. Sagan, respectively, showed that the non-linear operator $a_k \mapsto a_k^2 - a_{k-1}a_{k+1}$ (cf. Notation 36) acting on the coefficients of real polynomials of the form $\sum_{k=0}^{n} a_k x^k$, all of whose zeros have non-positive real parts, yields a polynomial of the same type ([9, Proof of Conjecture 1.1]). In particular, this operator preserves polynomials belonging to the Laguerre-Pólya class $\mathscr{L}\text{-}\mathscr{P}^+$, and moreover, the same result holds for transcendental functions in $\mathscr{L}\text{-}\mathscr{P}^+$ ([9, Theorem 5.4]). In [43], the above non-linear operator was extended to a class of non-linear operators (see Definition 40) that preserve $\mathscr{L}\text{-}\mathscr{P}^+$. Remarkably, this class of non-linear operators is defined in terms of the coefficients of a system of inequalities that characterizes the Laguerre-Pólya class $\mathscr{L}\text{-}\mathscr{P}$.

The primary objective of this work is to call attention to the zero-preserving properties of non-linear coefficient-wise transformations. While a complete characterization of non-linear stability and hyperbolicity preservers remains a mystery, we will show that non-linear coefficient-wise operations play an important role in the classical theory of stability.

1.3.1 Notions of stability

An elementary question in the study of the distribution and location of zeros of functions, and one that is surprisingly difficult to settle, deals with determining when do the real parts of the zeros of a function all have the same sign.

Definition 2. A function is called *stable* (*Hurwitz stable*) if all its zeros lie in the open left half-plane and it is called *weakly stable* (*weakly Hurwitz stable*) if all its zeros lie in the closed left half-plane. □

Recently, it has become convenient to define stability by where the zeros are not to be found.

The advantage of Definition 3 over Definition 2 becomes apparent when working in the multivariate case.

Definition 3. A function $f(z)$ is stable with respect to the region Ω if $f(z) \neq 0$ for $z \in \Omega$, and we call such a function Ω-stable. $\square$

Thus, a weakly Hurwitz stable polynomial is open-right-half-plane-stable and a Hurwitz stable polynomial is closed-right-half-plane-stable. We shall always use Definition 2 when working with univariate stability.

A foundational result of stability provides a necessary condition for the Hurwitz stability of a real polynomial. The following theorem is of a folklore origin and is sometimes attributed to the great Slovakian engineer A. B. Stodola ([40, p.204], [50]).

Theorem 4 (A. B. Stodola, 1893). *If a real polynomial is Hurwitz stable, then all its coefficients are positive.*

Proof. Let $r_1, r_2, \ldots, r_n$, be the real zeros of a real polynomial $p(x)$ and let $\alpha_1, \alpha_2, \ldots, \alpha_m$, be its non-real zeros. Then, by the Fundamental Theorem of Algebra [42], we have the factorization of $p(x)$,

$$\begin{aligned} p(x) &= \prod_{k=1}^{n}(x-r_k)\prod_{j=1}^{m}(x-\alpha_j)(x-\overline{\alpha_j}) \\ &= \prod_{k=1}^{n}(x-r_k)\prod_{j=1}^{m}(x^2-(2\Re\alpha_j)x+|\alpha_j|^2). \end{aligned} \tag{1.2}$$

Now, as $p(x)$ is assumed to be Hurwitz stable, $r_k < 0, k = 1, 2, \ldots, n$, and $\Re\alpha_j < 0, j = 1, 2, \ldots, m$, and this renders all the constants in (**??**) positive. $\square$

Remark 5. The theory of stability would be much less exciting than the proof of Theorem 4 were its converse true. Consider for example $x^2 - x + 1$, whose zeros lie in the open right half-plane, while $(x^2 - x + 1)(x + 1)(x + 2)^2 = x^5 + 4x^4 + 4x^3 + x^2 + 4x + 4$.

The program initiated by A. Stodola and A. Hurwitz culminated in the celebrated Routh-Hurwitz Theorem (see for example [40, Chapter XV], [44, 15.715], [50], [83]), a criterion for determining stability. Other frequently used results in stability include the Hermite-Biehler Theorem (see [80, p. 197]), the Routh-Hurwitz criterion (see [48]), and the methods of continued fractions (see [49]). Composition theorems such as the Malo-Schur-Szegö Theorem (see [22], [67, §16], [71, §7]) yield sharper results on the stability of polynomials, especially when considering polynomials all of whose zeros lie in a sector. However, results distilled from the application of Sturm's Theorem (see [52, Vol. III], [71, §18]), which is inapplicable to the non-real zeros of a polynomial, do not apply in the analysis of stability. For a survey of sector preservers we refer to [14] and the references contained therein.

We conclude this section by attempting to collect and reconcile the terminology found throughout the vast literature on the subject of stability.

A real polynomial $p(x)$ is sometimes called:

(i) *elliptic* if all its roots are non-real ([4]),

(ii) *sinusoidal* if all its zeros are purely imaginary or 0 ([41]),

(iii) *almost sinusoidal* if exactly one zero is not purely imaginary or 0, and is negative ([41]),

(iv) *Schur stable* if all its zeros lies in the open disk $\{z : |z| < 1\}$ ([41]),

(v) *standard* if its leading leading coefficient is positive ([41]),

(vi) *aperiodic* if all its zeros are simple and negative ([41]),

(vii) *positive* if $p(x) > 0$ for all $x \in \mathbb{R}$ ([4]),

(viii) *non-negative* if $p(x) \geq 0$ for all $x \in \mathbb{R}$ ([4]),

(ix) *a sum-of-squares* if $p(x) = \sum_{k=0}^{n} p_k^2(x)$ for some sequence of real polynomials $\{p_k(x)\}_{k=0}^{n}$ ([4], [90, p. 132]),

(x) *sign-independently real-rooted* if all the its roots are real and also all the roots of all polynomials obtained by means of arbitrary sign changes of the coefficients are real ([72]).

The (Hurwitz) stable polynomials are sometimes also called *asymptotically* stable ([41]), or *Gårding stable* ([6], [39]). The weakly (Hurwitz) polynomials have been referred to as *quasi-stable* ([41]). Now, a real polynomial $p(x)$ all of whose zeros are real is most commonly referred to as *hyperbolic*, due to the terminology associated with equations studied by L. Gårding.

1.4 The Laguerre-Pólya class

We list here some of the properties of the the Laguerre-Pólya class and refer to [65, Ch. VIII], [71], [75], and the references contained therein for a detailed study. As we are about to list several equivalent formulations of the class of entire functions commonly known as the Laguerre-Pólya class, we take the following as a starting definition.

Definition 6 ([65, Ch. VIII], [71, p.10], [74, p. 54], [74, p. 105, Satz I]). A real entire function belongs to the Laguerre-Pólya class if and only if it is the uniform limit, on compact subsets of $\mathbb{C}$, of real polynomials all of whose zeros are real. □

A real entire function $\psi(x) = \sum_{k=0}^{\omega} \frac{\gamma_k}{k!} x^k, 0 \leq \omega \leq \infty$, is the local uniform limit of real polynomial all of whose zeros are real if and only if $\psi(x)$ can be expressed in the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-\dfrac{x}{x_k}} \qquad (0 \leq \omega \leq \infty) \,, \tag{1.3}$$

where $m \in \mathbb{N}$, $\alpha, \beta, c, x_k \in \mathbb{R}$, $\alpha \geq 0$, and $\sum_{k=1}^{\omega} \frac{1}{x_k^2} < \infty$. By convention, we permit $\omega = 0$ and define the empty product to be 1. The Laguerre-Pólya class is typically denoted by $\mathscr{L}$-$\mathscr{P}$, but due to the connection with *multiplier sequences of the second kind* (cf. Theorem 18) the symbol $\mathscr{L}$-$\mathscr{P}$II is also used and a function $\psi(x)$ with the product representation (1.3) is said to be of *type II in the Laguarre-Pólya class*. If all the zeros of $\psi(x) \in \mathscr{L}$-$\mathscr{P}$ lie in the interval (a, b) (or $[a, b]$) for some $-\infty \leq a < b \leq \infty$, we will write $\psi(x) \in \mathscr{L}$-$\mathscr{P}\,(a, b)$ (or $\psi(x) \in \mathscr{L}$-$\mathscr{P}\,[a, b]$). If $-\gamma_k \geq 0$ or $(-1)^k \gamma_k \geq 0$ for all $k = 0, 1, 2, \ldots$, then $\psi(x) \in \mathscr{L}$-$\mathscr{P}$ is said to be of *type I in the Laguerre-Pólya class* and, due to the connection with *multiplier sequences of the first kind*, we will write $\psi(x) \in \mathscr{L}$-$\mathscr{P}$I (cf. Theorem 17). An important subclass of the Laguerre-Pólya class, denoted $\mathscr{L}$-$\mathscr{P}^+$ (or $\mathscr{L}$-$\mathscr{P}$I$^+$), consists of precisely those $\varphi(x) \in \mathscr{L}$-$\mathscr{P}$ whose Taylor coefficients are non-negative. A function $\varphi(x)$ belongs to $\mathscr{L}$-$\mathscr{P}^+$ if and only if $\varphi(x)$ can be expressed in the form

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) \qquad (0 \leq \omega \leq \infty) \,, \tag{1.4}$$

where $m \in \mathbb{N}$, $\sigma, c, x_k \in \mathbb{R}$, $\sigma, c \geq 0$, $x_k > 0$, and $\sum_{k=1}^{\omega} \frac{1}{x_k} < \infty$.

It is clear that $\mathscr{L}$-$\mathscr{P}$I$^+$ = $\mathscr{L}$-$\mathscr{P}$I$(-\infty, 0]$. If $\varphi \in \mathscr{L}$-$\mathscr{P}$I, then either $\varphi \in \mathscr{L}$-$\mathscr{P}(-\infty, 0]$ or $\varphi \in \mathscr{L}$-$\mathscr{P}[0, \infty)$; however, by considering the function $\frac{1}{\Gamma(x)}$, where $\Gamma(x)$ denotes the gamma function (see [81, §8]), we see that functions belonging to $\mathscr{L}$-$\mathscr{P}(-\infty, 0]$ need not belong to $\mathscr{L}$-$\mathscr{P}$I.

The Laguerre-Pólya class can be characterized by means of systems of inequalities. We list here the known systems of inequalities that characterize the Laguerre-Pólya class, the principal of which are the *extended Laguerre inequalities.*

Theorem 7 (Laguerre inequality: Real version. [18], [21], [73])**.** *Let $f(x)$ be a real entire function of genus 0 or 1 and let $\varphi(x) = e^{-\alpha x^2} f(x)$, where $\alpha \geq 0, \varphi(x) \not\equiv 0$. Define*

$$L_p(\varphi(x)) := \sum_{j=0}^{2p} \frac{(-1)^{p+j}}{(2p)!} \binom{2p}{j} \varphi^{(j)}(x)\varphi^{(2p-j)}(x) \ , \tag{1.5}$$

where $x \in \mathbb{R}$ and $p = 0, 1, 2, \ldots$. Then $\varphi(x) \in \mathscr{L}$-$\mathscr{P}$ if and only if for all $x \in \mathbb{R}$ and all $p = 0, 1, 2, \ldots$

$$L_p(\varphi(x)) \geq 0 \ . \tag{1.6}$$

$\square$

Observe that $L_1(\varphi(x)) = (\varphi'(x))^2 - \varphi(x)\varphi''(x)$, and $L_1(\varphi(x)) \geq 0$ is sometimes called the *Laguerre inequality* (for example in [18], [43]). In Theorem 13, $L_1(\varphi(x))$ is used to obtain another characterization of the Laguerre-Pólya class. D. Cardon generalized the extended Laguerre inequalities and obtained a novel characterization of the Laguerre-Pólya class. The work of D. A. Cardon [11] generalizes Theorem 7 and, for our purposes in particular, provides a new tool with which to study the sequences of complex numbers that define the class of non-linear operators of Definition 40.

Theorem 8 (Cardon's Theorem. [11])**.** *Let $f(z) := e^{-\alpha z^2}\phi(z)$ be not identically zero, where $\phi(z)$ is a real entire function of genus 0 or 1 and $\alpha \geq 0$. Let $g(z) = \prod_{k=1}^{m}(z + \alpha_k)$ be an even polynomial with non-negative real coefficients and at least one non-real zero. Then, $f(z) \in \mathscr{L}$-$\mathscr{P}$ if and only if for all $z \in \mathbb{R}$ and all non-negative integers k,*

$$A_k(z) := \frac{1}{k!} \left[\frac{d^k}{dt^k} \prod_{j=1}^{m} f(z + \alpha_j t) \right] \Bigg|_{t=0} \geq 0 \ . \tag{1.7}$$

$\square$

Setting $g(z) = z^2+1$ in Theorem 8 produces Theorem 7. For this reason we regard the inequalities in (1.7) (cf. (3.13)) as generalizations of the extended Laguerre inequalities of Theorem 7 and we will sometimes refer to the expressions in (1.7) as the *Cardon expressions* (see §3.2). Next, we have complex analogues of the Laguerre inequalities.

Theorem 9 (Laguerre inequality: Complex version I. [30])**.** *Let $f(z)$ be an entire function of the form (1.3) and let the zeros $\{z_k\}_{k=1}^{\infty}$ of $f(z)$ be counted according to multiplicity and arranged so that $0 \leq |z_1| \leq |z_2| \leq |z_3| \leq \cdots$. Then, $f(z) \in \mathscr{L}\text{-}\mathscr{P}$ if and only if for all $z := x+iy \in \mathbb{C}, y \neq 0$,*

$$\frac{1}{y}\Im\{-f'(z)\overline{f(z)}\} \geq 0 \ . \tag{1.8}$$

□

Theorem 10 (Laguerre inequality: Complex version II. [53])**.** *Let $f(z) := e^{-\alpha z^2}\phi(z)$ be not identically zero, where $\phi(z)$ is a real entire function of genus 0 or 1 and $\alpha \geq 0$. Then, $f(z) \in \mathscr{L}\text{-}\mathscr{P}$ if and only if for all $z \in \mathbb{C}$,*

$$|f'(z)|^2 \geq \Re\{f(z)\overline{f''(z)}\} \ . \tag{1.9}$$

□

Theorem 11 ([18, Theorem 2.7])**.** *Let $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!}x^k$ denote a real entire function, let $g_n(t) := \sum_{k=0}^{n}\binom{n}{k}\gamma_k t^k, n = 0,1,2,\ldots,$ and let $\Delta_n(t) := g_n^2(t) - g_{n-1}(t)g_{n+1}(t), n = 1,2,3,\ldots$. Then $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$ if and only if the following conditions hold:*

(i) $\Delta_n(t) \geq 0$ *for all positive integers n and all $t \in \mathbb{R}$.*

(ii) *If $\gamma_0 \neq 0$ and $\gamma_1^2 - \gamma_0\gamma_2 > 0$, then*

  (a) $g_{n+1}(t_0) = 0$, *whenever* $\Delta_n(t_0) = 0, t_0 \neq 0$, *and*

  (b) $\gamma_{n+1} = 0$, *whenever* $\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} = 0$.

(iii) *If $\gamma_0 \neq 0$ and $\gamma_1^2 - \gamma_0\gamma_2 = 0$, then $\varphi(x) = \gamma_0 e^{\frac{\gamma_1}{\gamma_0}x}$.*

(iv) *If $\gamma_0 = 0$, then $\varphi(x) = x^r\psi(x)$, with $\psi(0) \neq 0$, where $\psi(x)$ satisfies (i), (ii) and (iii) for the appropriately redefined γ_n, g_n and Δ_n.*

□

The polynomial $g_n(x) = \sum_{k=0}^{n}\binom{n}{k}\gamma_k x^k, n = 0,1,2,\ldots$, associated to a real entire function $\varphi(x) = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k$, is called the n^{th} *Jensen polynomial* associated to $\varphi(x)$. Jensen polynomials are the natural approximating polynomials for the Laguerre-Pólya class. If $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$, then the sequence $\{g_n\left(\frac{x}{n}\right)\}_{n=0}^{\infty}$ approximates $\varphi(x)$ locally uniformly (see for example [18, Lemma 2.2]).

The next result comes from the theory of *total positivity* (see Definition 27).

Theorem 12 ([2], [35]). *Let $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k$ be a real entire function. Define the 4-way infinite matrix $A = (a_{ij})$ by setting $a_{ij} = a_{k+(i-j)}$. Then $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}^+$ if and only if all minors of all orders of A are non-negative.* □

For the next characterization of the Laguerre-Pólya class it is useful to begin with functions whose zeros all lie in a the closed strip

$$\overline{S}(A) = \{z \in \mathbb{C} \ : \ \Im|z| \leq A\} \ . \tag{1.10}$$

Let the class $\mathfrak{S}(A), 0 \leq A < \infty$, of entire functions consist of functions of the form (1.3) with $x_k \in \overline{S}(A) \setminus \{0\}$. Thus, if $f \in \mathfrak{S}(A)$, for some $A \geq 0$, has only real zeros, then $f \in \mathscr{L}\text{-}\mathscr{P}$.

Theorem 13 ([28, p.344]). *Let $f(z) \in \mathfrak{S}(A)$. For $\mu \in \mathbb{R} \setminus \{0\}$ define $f_\mu(x) := 2\cos(\mu D) f(x)$, where $D = \frac{d}{dx}$. Then $f(x) \in \mathscr{L}\text{-}\mathscr{P}$, that is $A = 0$, if and only if for all $x, \mu \in \mathbb{R}, \mu \neq 0, L_1(f_\mu(x)) \geq 0$ and*

$$\left(\Im[f'(x+i\mu)]\right)^2 - \Im[f(x-i\mu)]\Im[f(x+i\mu)] \geq 0 \ . \tag{1.11}$$

□

Theorem 14 ([71, Satz 21.3]). *Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial with zeros $\alpha_1, \ldots, \alpha_n$, counted according to multiplicity. Let $S_m = \sum_{k=0}^{n} a_k^m$. Then $p \in \mathscr{L}\text{-}\mathscr{P}$ if and only if the quadratic form*

$$U_m = \sum_{k=1}^{n}\sum_{j=1}^{n} S_{k+j-2} x_k x_j \ , \tag{1.12}$$

is positive for all $x_k, x_j \in \mathbb{R}$, or equivalently, each determinant in the sequence

$$S_0 = n, \begin{vmatrix} S_0 & S_1 \\ S_1 & S_2 \end{vmatrix}, \begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix}, \ldots, \begin{vmatrix} S_0 & S_1 & S_2 & \cdots & S_{n-1} \\ S_1 & S_2 & S_3 & \cdots & S_n \\ S_2 & S_3 & S_4 & \cdots & S_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{n-1} & S_n & S_{n+1} & \cdots & S_{2n-2} \end{vmatrix}, \tag{1.13}$$

where S_m is the discriminant of U_m, is non-negative. Moreover, the zeros of p are all real and simple if and only if each determinant in the above sequence is positive. $\square$

In the sequel (see §3.2) we collect some of the known necessary but not sufficient conditions for membership in the Laguerre-Pólya class. As infinitely many necessary conditions for membership in the Laguerre-Pólya class may be obtained by means of the theorems appearing in this section, we feel that the tedious search for necessary conditions that are not sufficient for membership in the Laguerre-Pólya class would not be rewarded either with amusement or with instruction.

1.4.1 Multiplier and complex zero decreasing sequences

In general, a sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is called a *multiplier sequence* if it acts diagonally on the monomial basis of $\mathbb{R}[x]$ and preserves $\mathscr{L}$-$\mathscr{P}$. We follow Pólya-Schur in distinguishing between two kinds of multiplier sequences, though various other *kinds* of multiplier sequences have since been introduced (see Remark 23).

Definition 15 (Multiplier sequence. [19, Definition 1.2])**.** A sequence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence* (*of the first kind*) if, whenever the real polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ has *only* real zeros, then the polynomial $\Gamma[p(x)] := \sum_{k=0}^{n} \gamma_k a_k x^k$ also has *only* real zeros. If $p(x)$ is a real polynomial with all real zeros of the same sign, then Γ is called a *multiplier sequence* (*of the second kind*) if the polynomial $\Gamma[p(x)]$ has *only* real zeros. $\square$

Multiplier sequences do preserve the Laguerre-Pólya class, and are therefore the first example of hyperbolicity preserving operators we have encountered (see Theorem 25). The order and type of $\mathscr{L}$-$\mathscr{P}$ is also invariant under application of a multiplier sequence.

Theorem 16 ([65, p. 343])**.** *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence. If $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}$-$\mathscr{P}$, then the function $\Gamma[f(x)] = \sum_{k=0}^{\infty} a_k \gamma_k x^k$ is entire and belongs to $\mathscr{L}$-$\mathscr{P}$.* $\square$

Later, in Corollary 80 we provide an analogue to Theorem 16 for a different type of non-linear coefficient-wise transformations.

In their seminal 1914 paper, G. Pólya and I. Schur completely characterized all multiplier sequences.

Theorem 17 (Characterization of multiplier sequences (of the first kind). [65, Chapter VIII], [71, Kapitel II], [75])**.** *Let $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers that acts diagonally on the monomial basis of $\mathbb{R}[x]$. Then the following are equivalent.*

(i) Γ *is a multiplier sequence of the first kind.*

(ii) (*Transcendental Characterization*) *The function*

$$\varphi(x) := \Gamma[e^x] = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \tag{1.14}$$

is entire and either $\varphi(x), \varphi(-x), -\varphi(x)$, *or* $-\varphi(-x)$ *belongs to* $\mathscr{L}\text{-}\mathscr{P}^+$.

(iii) (*Algebraic Characterization*) *For each* $n = 0, 1, 2, \ldots,$

$$\Gamma[(1+x)^n] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+ . \tag{1.15}$$

□

Theorem 18 (Characterization of multiplier sequences (of the second kind). [65, Chapter VIII], [71, Kapitel II], [75])**.** *Let* $\Gamma := \{\gamma_k\}_{k=0}^{\infty}, \gamma_0 \neq 0$, *be a sequence real numbers that acts diagonally on the monomial basis of* $\mathbb{R}[x]$. *Then the following are equivalent.*

(i) Γ *is a multiplier sequence of the second kind.*

(ii) (*Transcendental Characterization*)

$$\Gamma[e^x] = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P} . \tag{1.16}$$

(iii) (*Algebraic Characterization*) *For each* $n = 0, 1, 2, \ldots,$

$$\Gamma[(1+x)^n] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \in \mathscr{L}\text{-}\mathscr{P} . \tag{1.17}$$

□

Remark 19. Multiplier sequences of the first kind acting on polynomials, all of whose zeros are real and of the same sign, preserve the reality and the sign of the zeros (cf. Theorem 25). In contrast, this is not true for multiplier sequences of the second kind, as can be seen by considering the sequence $\{-1 + k + k^2\}_{k=0}^{\infty}$ generated by $(x^2 + 2x - 1)e^x$, and the polynomial $(x+1)^2$.

Definition 20 (Complex zero decreasing sequence. [19, Definition 1.3])**.** We say that a sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a *complex zero decreasing sequence* (CZDS) if

$$Z_c\left(\sum_{k=0}^{n}\gamma_k a_k x^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_k x^k\right) \tag{1.18}$$

for any real polynomial $p(x)=\sum_{k=0}^{n} a_k x^k$, where $Z_c(p(x))$ denotes the number of non-real zeros of $p(x)$, counting multiplicities. The acronym CZDS will also be used in the plural. □

The next theorem is a classical result of E. Laguerre and a foundational result in the theory of stability as it asserts the existence of non-trivial CZDS.

Theorem 21 ([71, Satz 3.2 (E. Laguerre)])**.**

(i) Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial and let $h(x) \in \mathscr{L}\text{-}\mathscr{P}$ be a polynomial none of whose zeros lie in the interval $(0, n)$. Then $\{h(k)\}_{k=0}^{n}$ is a CZDS.

(ii) Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial and let $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$ be a transcendental function none of whose zeros lie in the interval $(0, n)$. Then $\{\varphi(k)\}_{k=0}^{n}$ is a CZDS.

(iii) $\{\varphi(k)\}_{k=0}^{\infty}$ is a CZDS, whenever $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}\ (-\infty, 0]$.

□

T. Craven and G. Csordas have classified all CZDS that can be interpolated by real polynomials. The complete characterization of CZDS remains a mystery.

Theorem 22 ([19, Theorem 2.13])**.** *Let $h(x)$ be a real polynomial. The sequence $\{h(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence if and only if either*

(i) $h(0) \neq 0$ and all the zeros of h are real and negative, or

(ii) $h(0) = 0$ and $h(x) = x(x-1)(x-2)\cdots(x-m+1)\prod_{i=1}^{p}(x-b_i)$, where m is a positive integer and $b_i < m$ for each $i = 1, 2, \ldots, p$.

□

Remark 23. It is almost unnecessary to remark that a CZDS is automatically a multiplier sequence. By Laguerre's Theorem, if $h(x) \in \mathscr{L}\text{-}\mathscr{P}^{+}$ is a polynomial, then the sequence $\{h(k)\}_{k=0}^{\infty}$ is a multiplier sequence of the first kind. By means of an example, we see that a multiplier sequence need not be a CZDS. For instance, $\{1+k+k^2\}_{k=0}^{\infty}$ is generated by $(x+1)^2e^x$, hence by (ii) of Theorem 17 it is a multiplier sequence of the first kind, but by Theorem 22 it is not a CZDS.

Multiplier sequences continue to enjoy the attention of researchers, in part due to the fact that Theorem 21, a classical result of E. Laguerre asserting the existence of non-trivial CZDS, provides the *only* known examples of complex zero decreasing sequences (cf. [6, Problem 20], [25, Problems 8 and 9]), and in part due to the connection with the celebrated Riemann Hypothesis (see [25, §4]).

A recent paper introduced *multiplier sequences of the third kind* (see [72]), which are defined to be the sequences that take sign-independently real-rooted polynomials with all non-negative coefficients into polynomials all of whose zeros are real. It turns out that *log-concavity* is a necessary and sufficient condition for a sequence of positive numbers to be a multiplier sequence of the third kind (cf. (iii) of Proposition 24).

For convenience, we list some properties of classical multiplier sequences that will be useful in the sequel.

Proposition 24 ([22], [65, p. 341])**.** *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence. Then*

(i) The sequence $\{\gamma_k\}_{k=m}^{\infty}$, where m is any non-negative integer, is also a multiplier sequence.

(ii) For any $r \in \mathbb{R} \setminus \{0\}$, the sequence $\{r\gamma_k\}_{k=0}^{\infty}$ is also a multiplier sequence.

(iii) The elements of $\{\gamma_k\}_{k=0}^{\infty}$ satisfy the Turán inequality

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, k = 1, 2, 3, \ldots. \tag{1.19}$$

(iv) The sequence $\{(-1)^k\gamma_k\}_{k=0}^{\infty}$ is also a multiplier sequence.

(v) The elements of $\{\gamma_k\}_{k=0}^{\infty}$ are all of the same sign, or they alternate in sign.

□

The notation associated with the *Laguerre-Pólya class* is often used to denote hyperbolic polynomials (cf. §1.4) and so we will use the following conventions throughout the sequel.

(i) If all the zeros of $p(x) \in \mathbb{R}[x]$ are real, then we write $p(x) \in \mathscr{L}\text{-}\mathscr{P}$.

(ii) If all the zeros of $p(x) \in \mathbb{R}[x]$ are real and of the same sign, then we write $p(x) \in \mathscr{L}\text{-}\mathscr{P}^+$.

(iii) The symbols $\mathscr{L}\text{-}\mathscr{P}_n$ and $\mathscr{L}\text{-}\mathscr{P}_n^+$ are sometimes used in the above cases when we are referring to $p(x) \in \mathbb{R}_n[x]$, the linear subspace of $\mathbb{R}[x]$ consisting of all real polynomials of degree at most n (cf. [27]).

In general, a polynomial $p(x)$ with not all real zeros may be written as a product of polynomial $g(x) \in \mathscr{L}\text{-}\mathscr{P}_n$ and an elliptic polynomial $f(x)$. The class of such polynomials is sometimes denoted as $\mathscr{L}\text{-}\mathscr{P}^*$ (or as $\mathscr{L}\text{-}\mathscr{P}_n^*$ if there is a restriction on the degree). The *Laguerre-Pólya class* $\mathscr{L}\text{-}\mathscr{P}^*$ (see §1.4) is the uniform limit, on compact subsets of $\mathbb{C}$, of real polynomials belonging to $\mathscr{L}\text{-}\mathscr{P}^*$. In general, $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}^*$ is the product of a an elliptic polynomial $p(x)$ and a function $\psi(x) \in \mathscr{L}\text{-}\mathscr{P}$.

1.5 A word concerning linear stability preserving operators

We have already encountered a class of stability preserving operators.

Theorem 25 ([17]). *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a non-negative multiplier sequence and let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a stable polynomial. Then $\Gamma[p(x)] = \sum_{k=0}^{n} a_k \gamma_k x^k$ is a stable polynomial.*

In light of the next theorem, which gives an explicit construction for linear stability preserving operators and the previous results on hyperbolic polynomials, it seems surprising that the complete characterization of such objects has evaded mathematicians for over a century.

Theorem 26 ([14, Theorem 205], [79, Proposition 29]). *Let $T : \mathbb{C}[x] \to \mathbb{C}[x]$ be a linear operator. Then there exists a unique set of polynomials $\{p_k(x)\}_{k=0}^{\infty}$, such that for all $f(x) \in \mathbb{C}[x]$,*

$$T[f(x)] = \sum_{k=0}^{\infty} p_k(x) f^{(k)}(x) \ . \tag{1.20}$$

□

Quite recently, J. Borcea and P. Brändén obtained a complete characterization of all linear operators that preserve stability of univariate and multivariate polynomials (see [5]). In retrospect, Theorem 26 is nice, but it's multivariate analogue (see for example [14, Theorem 205]) does much more to motivate the characterization problem of linear stability preservers than all univariate classical results in stability theory.

1.6 Classical non-linear stability preservers

1.6.1 Index shifts in totally positive sequences

According to the theory of *totally positive* sequences developed by M. Aissen, A. Edrei, I. J. Schoenberg, and A. M. Whitney ([2], [35]), the coefficients $\{a_k\}_{k=0}^{\infty}$ of a function $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+$ form a totally positive sequence. In fact, more is true as we have seen in Theorem 12 that total positivity is intimately related to the Laguerre-Pólya class.

Definition 27. A sequence of positive numbers $\{a_k\}_{k=0}^{\infty}$ is called *totally positive of order* m, denoted TP_m, if all minors of all orders up to m of the 4-way infinite matrix $A = (a_{ij}), a_{ij} = a_{i-j}$, are positive. If the sequence is TP_m for all positive integers m, then we call it a *totally positive* sequence and write TP_∞ or TP. The matrix A is called the *Toeplitz* matrix associated to $\{a_k\}_{k=0}^{\infty}$ (cf. (3.28)). $\square$

Consider the 4-way infinite Toeplitz matrix $A = (a_{ij})$, obtained from the coefficients of the real function $\varphi(z) = \sum_{k=0}^{\omega} a_k z^k \in \mathscr{L}\text{-}\mathscr{P}^+, 0 \le \omega \le \infty$, by setting $a_{ij} = a_{k+(i-j)}$. Suppose first that $\omega = m < \infty$, so that $\varphi(z)$ is a real polynomial of degree m. Let n, ℓ, q, be non-negative integers such that $0 \le n \le \ell$ and $n + q\ell \le m \le n + (q+1)\ell$. We obtain the a submatrix $B = (b_{ij})$ of A by setting $b_{ij} = a_{k+\ell(i-j)}$. The matrix B is totally positive and, consequently, the zeros of the polynomial

$$\sum_{k=0}^{q} a_{n+k\ell} z^k \tag{1.21}$$

are all real and negative. The extension to transcendental entire functions is immediate (see also [76, V, 171.4]).

Theorem 28 ([2])**.** *If* $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathscr{L}\text{-}\mathscr{P}^+$*, and* n, ℓ *are non-negative integers such that* $0 \le n < \ell$*, then* $\sum_{k=0}^{\infty} a_{n+k\ell} z^k \in \mathscr{L}\text{-}\mathscr{P}^+$*.* $\square$

Corollary 29. *If* $a_k \mapsto \Lambda_k$ *is a non-linear stability preserving operator, then so is* $a_k \mapsto \Lambda_{n+k\ell}$*, where* n, ℓ *are non-negative integers such that* $0 \le n < \ell$*.* $\square$

We remark that the properties of classical multiplier sequences are in a certain sense inherited by non-linear stability preserving operators. A multiplier sequence $\{\gamma_k\}_{k=0}^{\infty}$ of the first kind is mapped to a multiplier sequence of the first kind $\{\Lambda^k\}_{k=0}^{\infty}$ by means of the non-linear stability preserving operator $a_k \mapsto \Lambda^k$.

1.6.2 The Hadamard product

Definition 30. Given real polynomials $p(x) = \sum_{k=0}^{n} a_k x^k$ and $q(x) = \sum_{k=0}^{m} b_k x^k$ the polynomial $p \star q(x) := \sum_{k=0}^{\min\{n,m\}} a_k b_k x^k$ is called the *Hadamard product* of p and q. $\square$

Given $p, r, s \in \mathbb{R}[x]$, and $\alpha, \beta \in \mathbb{R}$, a routine calculation verifies the linearity of the Hadamard product; i.e., $p(x) \star [\alpha r(x) + \beta s(x)] = \alpha p \star r(x) + \beta p \star s(x)$. Operating on the coefficients of a polynomial $\sum_{k=0}^{n} a_k x^k$, however, the operation $a_k \mapsto a_k^2$, is *quadratic* and we therefore regard $\star$ as a non-linear coefficient-wise operator.

The Hadamard product of stable polynomials is known to be stable (see [41]), so the class of non-linear stability preserving operators together with the Hadamard product should form a semi-group. The next proposition makes this notion precise.

Proposition 31. *If $a_k \mapsto \Lambda_k^1$ and $a_k \mapsto \Lambda_k^2$ are two non-linear operators that preserve real stable polynomials, then so is the non-linear operator $a_k \mapsto \Lambda_k^1 \Lambda_k^2$.*

Proof. Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a real stable polynomial. Then the polynomials $\sum_{k=0}^{n} \Lambda_k^1 x^k$ and $\sum_{k=0}^{n} \Lambda_k^2 x^k$ are also stable, as is their Hadamard product, $\sum_{k=0}^{n} \Lambda_k^1 \Lambda_k^2 x^k$. $\square$

Extending this property to functions that are locally uniformly approximated by polynomials may be accomplished by the application of the next result and a well-known theorem of P. Montel (see [16, p. 153], [70]).

Lemma 32 ([84])**.** *Let $\{P_n(z)\}_{n=0}^{\infty}$ be a sequence of polynomials, where*

$$P_n(z) = \sum_{k=0}^{M_n} a_{n,k} z^k, a_{n,0} \neq 0, a_{n,M_n} \neq 0, M_n \to \infty, \tag{1.22}$$

all of whose zeros lie in an open half-plane $H \subset \mathbb{C}$ with boundary containing the origin. If for some constants α_0, α_1, and all n,

$$0 < \alpha_0 \leq |a_{n,0}| \leq \alpha_1, \ |a_{n,1}| \leq \alpha_1, \ |a_{n,2}| \leq \alpha_1 < \infty \ , \tag{1.23}$$

then the sequence $\{P_n(z)\}_{n=0}^{\infty}$ is uniformly bounded in any circle $|z| \leq r$, and in fact,

$$|P_n(z)| \leq \alpha_1 \exp\left(r\frac{\alpha_1}{\alpha_0} + 3r^2 \left(\frac{\alpha_1^2}{\alpha_0^2} + \frac{\alpha_1}{\alpha_0} \right) \right) \ . \tag{1.24}$$

$\square$

Theorem 33. *If $\Lambda^1 : a_k \mapsto \Lambda^1_k$ and $\Lambda^2 : a_k \mapsto \Lambda^2_k$ are two non-linear operators that preserve stable polynomials, then the non-linear operator $a_k \mapsto \Lambda^1\Lambda^2$ preserves stable transcendental functions that are the limits, on compact subsets of $\mathbb{C}$, of real stable polynomials.*

Proof. Let $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k$ be a transcendental entire function that can be uniformly approximated, on compact subsets of $\mathbb{C}$, by a sequence $\{P_n(x)\}_{n=1}^{\infty}$ of real stable polynomials. Then, both the sequences $\{\Lambda^1[P_n(x)]\}_{n=1}^{\infty}$ and $\{\Lambda^2[P_n(x)]\}_{n=1}^{\infty}$ are sequences of real stable polynomials and, by Proposition 31, the sequence $\{\Lambda^1[\Lambda^2[P_n(x)]]\}_{n=1}^{\infty}$ is a sequence of real stable polynomials. Now, letting $P_n(x) = \sum_{k=0}^{n} b_k x^k$ we have $\Lambda^1[\Lambda^2[P_n(x)]] = \sum_{k=0}^{n} \Lambda^1_k \Lambda^2_k x^k = \sum_{k=0}^{n} a_{n,k} x^k$. By Theorem 4, the coefficients $a_{n,k}$ are all positive, and therefore there exist positive constants α_0 and α_1 satisfying (1.23). By Lemma 32, the sequence $\{\Lambda^1[\Lambda^2[P_n(x)]]\}_{n=2}^{\infty}$ is locally uniformly bounded on compact subsets of $\mathbb{C}$, and by Montel's Theorem there exists a subsequence of $\{\Lambda^1[\Lambda^2[P_n(x)]]\}_{n=2}^{\infty}$, converging, uniformly on the compact subsets of $\mathbb{C}$, to the entire function $\Lambda^1[\Lambda^2[\varphi(x)]]$, which by Hurwitz's Theorem (Theorem 97) must also be stable. $\square$

1.6.3 Inversion

The left-half plane is invariant under the transformation $z \mapsto \frac{1}{z}$. Hence, if $p(x)$ is a real stable polynomial of degree n, then so is $x^n p(\frac{1}{x})$, and if the zeros of $p(x)$ are all real and of the same sign, the zeros of $x^n p(\frac{1}{x})$ will also be all real and of the same sign.

The superscript * is often used to denote inversion; i.e., $x^n p(\frac{1}{x}) := p^*(x)$, where $p(x)$ is a polynomial of degree n. Inversion's principal use is to reverse the order of the coefficients of a polynomial while leaving the real zeros fixed. This operation is deceptively pleasant. One must keep in mind the observation that inversion does not preserve the degree of a polynomial if the polynomial vanishes at 0.

Given $p, q \in \mathbb{R}[x]$, and $\alpha, \beta \in \mathbb{R}$, a routine verification shows that $(\alpha p(x) + \beta q(x))^* = \alpha p^*(x) + \beta q^*(x)$. Operating on the coefficients of $p(x) = \sum_{k=0}^{n} a_k x^k$, however, amounts to applying the coefficient-wise operator $a_k \mapsto a_{n-k}$, which is non-linear.

CHAPTER 2
A CLASS OF NON-LINEAR STABILITY PRESERVING OPERATORS

2.1 Introduction

Linear operators acting diagonally on the monomial basis of $\mathbb{C}[z]$ that preserve hyperbolicity (stability) have been completely characterized by G. Pólya and I. Schur in their seminal 1914 paper. These are the multiplier sequences of the first (second) kind. Little is known about non-linear coefficient-wise hyperbolicity and stability preservers and the only known examples of such operators are mentioned in Section 1.6. Nevertheless, the notion of a non-linear coefficient-wise hyperbolicity preserver may be traced back to a 1989 study of the relationship between the Laguerre and the Turán inequalities, in which T. Craven and G. Csordas have posed the following problem:

Problem 34 ([18]). *Classify the functions*

$$\psi(x) = \sum_{k=0}^{\omega} \frac{\gamma_k}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P}, \tag{2.1}$$

where $\gamma_k \geq 0$ and $0 \leq \omega \leq \infty$, for which the functions

$$f(x) := \sum_{k=0}^{\infty} \frac{\gamma_{k+1}^2 - \gamma_k \gamma_{k+2}}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P}. \tag{2.2}$$

□

Of course, Problem 34 is also valid various shifts in indices, so long as the usual conventions are followed (cf. Notation 36), and so the next result, conjectured independently by S. Fisk (see [36, Question 1]), R. P. Stanley, P. R. W. McNamara and B. E. Sagan (see [69]), and proved by P. Brändén, provides a partial answer to Problem 34.

Theorem 35 (Brändén's Theorem. [9]). *If the zeros of the real polynomial $\psi(x) = \sum_{k=0}^{n} a_k z^k$ are all real and negative, then the zeros of the polynomial*

$$\sum_{k=0}^{n} (a_k^2 - a_{k-1}a_{k+1}) z^k \,, \text{ where } a_{-1} := 0 \text{ and } a_{n+1} := 0 \,, \tag{2.3}$$

are all real and negative. □

We hasten to remark that Theorem 35 also applies to $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}^+$ (see Theorem 38). Brändén's Theorem, also known prior to its resolution as the Stanley conjecture, is a partial answer to Problem 34, which in turn has been motivated by a still unresolved conjecture of G. Boros and V. Moll and results on polynomials due to I. Newton (see §4.1). P. Brändén's result is the first step toward the characterization of non-linear coefficient-wise transformations that preserve hyperbolicity or that preserve stability. The complete characterization remains a mystery.

A notation scheme for non-linear transformations of sequences has already been introduced by P. Brändén in [9]. We adopt it here and at the same time set conventions that will useful throughout the sequel.

Notation 36.

(i) To a real sequence $\mu = \{\mu_j\}_{j=0}^{\infty}$ we associate the transformation of the real sequence $\{a_k\}_{k=0}^{\infty}$,

$$\Lambda_k^p(\mu) := \sum_{j=0}^{p} \mu_j \prod_{j=1}^{n} a_{\sigma_j(k,j)} \ , \tag{2.4}$$

typically taken to be the coefficients of a real entire function. In the absence of an explicit sequence μ, we will refer to the non-linear operator $\Lambda^p(\mu) : a_k \mapsto \Lambda_k^p(\mu)$ as the operator generated by the polynomial g, which generates μ in the sense of Theorem 8, and write $\Lambda^p(g)$.

(ii) We will denote the action of the non-linear operator $\Lambda^p(\mu) : a_k \mapsto \Lambda_k^p(\mu)$ on the polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ by $\Lambda^p(\mu)[p(x)] = \sum_{k=0}^{n} \Lambda_k^p(\mu)\, x^k$. We will also write Λ^p in place of $a_k \mapsto \Lambda_k^p$ when referring to the operator that replaces the coefficient a_k with the non-linear combination of coefficients defined by Λ_k^p, and similarly treat other operators appearing in the sequel.

(iii) Given a generating polynomial $g(z)$ of degree n, we will call an expression of the form (2.4) a non-linear coefficient-wise transformation of degree n and order p. The order of $\Lambda^p(g)$ refers to the number of terms in $\Lambda_k^p(g)$ and the degree refers to the number of factors in each term.

(iv) Throughout the sequel it will be convenient to extend sequences of coefficients to negative indices. We will follow the convention on the integer index k that $a_k = 0$, whenever k is non-positive for transcendental entire functions $\psi(x)$, and if $\psi(x)$ is a polynomial, then we will set $a_k = 0$, whenever $k \notin \{0, 1, 2, \ldots, \deg \psi(x)\}$.

(v) In order to avoid confusion, and for technical reasons alike, we will always explicitly write $a_k = \frac{\gamma_k}{k!}$ to distinguish between the coefficients a_k of $\psi(x) = \sum_{k=0}^{\infty} a_k x^k$ and the coefficients $\frac{\gamma_k}{k!}$ of $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$. This is necessary, because given a sequence of positive numbers $\{\frac{\gamma_k}{k!}\}_{k=0}^{\infty}$, the inequalities $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, k = 1, 2, 3, \ldots$, may all hold while some of the Turán-type expressions, $(a_k k!)^2 - a_{k-1}a_{k+1}(k-1)!(k+1)!$, may be negative. Consider, for example, the sequence defined by $\gamma_k = 2^k k!, k = 0, 1, 2, \ldots$. Indeed, $a_k^2 - a_{k-1}a_{k+1} \geq 0$ if and only if $\gamma_k^2 - \frac{k}{k+1}\gamma_{k-1}\gamma_{k+1} \geq 0$ (see §4.1).

□

P. Bränden characterized the hyperbolicity preserving transformations of sequences of coefficients $\{a_k\}_{k=0}^{\infty}$ given by $\Lambda^p(\alpha) : a_k \mapsto \Lambda_k^p(\alpha)$, where $\alpha = \{\alpha_j\}_{j=0}^{\infty}$ is a sequence of real numbers and $\Lambda_k^p(\alpha) = \sum_{j=0}^{p} \alpha_j a_{k-j} a_{k+j}$.

Theorem 37 ([9, Theorem 4.2])**.** *Let $\alpha = \{\alpha_k\}_{k=0}^{\infty}$ be a sequence of real numbers and let $\mathscr{L}\text{-}\mathscr{P}_n^+$ denote the class of real polynomials of degree at most n with only real and non-positive zeros. The following are equivalent.*

(i) $\Lambda(\alpha)[\mathscr{L}\text{-}\mathscr{P}_n^+] \subseteq \mathscr{L}\text{-}\mathscr{P}_n^+ \cup \{0\}$ *;*

(ii) $\Lambda(\alpha)[(z+1)^n] \subseteq \mathscr{L}\text{-}\mathscr{P}_n^+ \cup \{0\}$ *;*

(iii)
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{j=0}^{k} \frac{\alpha_j}{(k+j)!(k-j)!} \right) \frac{z^k}{(n-2k)!} \in \mathscr{L}\text{-}\mathscr{P}_n^+ \cup \{0\} .$$

□

Theorem 38 ([9, Theorem 5.7])**.** *Let $\alpha = \{\alpha_k\}_{k=0}^{\infty}$ be a sequence of real numbers. Let $\mathscr{L}\text{-}\mathscr{P}_n^+$ denote the class of real polynomials of degree at most n with only real and non-positive zeros and set $P^+ = \bigcup_{n=0}^{\infty} \mathscr{L}\text{-}\mathscr{P}_n^+$. The following are equivalent.*

(i) $\Lambda(\alpha)[P^+] \subseteq P^+ \cup \{0\}$ *;*

(ii) $\Lambda(\alpha)[e^z] \subseteq \mathscr{L}\text{-}\mathscr{P}^+ \cup \{0\}$*, that is,*

$$\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{\alpha_j}{(k-j)!(k+j)!} \right) z^k \in \mathscr{L}\text{-}\mathscr{P}^+ \cup \{0\}.$$

□

As a joint corollary of this characterization, we show that the sequences obtained form the coefficients of the extended Laguerre expressions of Theorem 7 satisfy Theorems 37 and 38. To proceed, we make precise this notation and terminology to be used throughout the sequel.

2.2 The class $\Lambda^p(z^2+1)$ of non-linear operators

A class of non-linear operators that act on the monomial basis of $\mathbb{C}[z]$ and preserve the Laguerre-Pólya class $\mathscr{L}\text{-}\mathscr{P}^+$ and weak Hurwitz stability has been introduced in [43] (see Theorem 42). It is remarkable that the system of inequalities that generalizes the Laguerre inequalities and characterizes the Laguerre-Pólya class $\mathscr{L}\text{-}\mathscr{P}$ (Theorem 7) also gives rise to a class of non-linear operators that preserves $\mathscr{L}\text{-}\mathscr{P}^+$. We focus in this section on the class of non-linear operators generated by the polynomial z^2+1 as described in Theorem 8.

Lemma 39 ([43])**.** *Let $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ be a real entire function and, for a positive integer p, let $L_p\left(\varphi^{(k)}(x)\right)$ be defined as in Theorem 7. Then,*

$$\frac{(2p)!}{2} \cdot L_p\left(\varphi^{(k)}(x)\right)\Big|_{x=0} = \binom{2p-1}{p}\gamma_{k+p}^2 + \sum_{j=1}^{p}(-1)^j\binom{2p}{p-j}\gamma_{k+p-j}\gamma_{k+p+j}\ . \tag{2.5}$$

Proof. Rewriting equation (1.5) with $\varphi^{(k)}(x)$, yields

$$L_p\left(\varphi^{(k)}(x)\right) = \sum_{j=0}^{2p}\frac{(-1)^{p+j}}{(2p)!}\binom{2p}{j}\varphi^{(k+j)}(x)\varphi^{(2p+k-j)}(x)\ . \tag{2.6}$$

For a fixed positive integer p, the coefficient of γ_{k+p}^2 is obtained by setting $j=p$ in the summand in (2.6),

$$\begin{aligned}\frac{(2p)!}{2}\cdot\frac{(-1)^{p+p}}{(2p)!}\binom{2p}{p}\varphi^{(k+p)}(0)\varphi^{(2p+k-p)}(0) &= \frac{1}{2}\binom{2p}{p}\gamma_{k+p}^2\\ &= \binom{2p-1}{p}\gamma_{k+p}^2\ .\end{aligned} \tag{2.7}$$

For a fixed $j = 1, 2, \ldots, p$, and an arbitrary positive integer p, the coefficient of $\gamma_{k+p-j}\gamma_{k+p+j}$ is obtained by setting $j = p-j$ or $j = p+j$ in the summand in (2.6). Thus, using the symmetry $\binom{2p}{p-j} = \binom{2p}{p+j}$,

$$2 \cdot \frac{(2p)!}{2} \cdot \frac{(-1)^{2p+j}}{(2p)!} \binom{2p}{p-j} \varphi^{(p+k-j)}(0)\varphi^{(p+k+j)}(0) = (-1)^j \binom{2p}{p-j} \gamma_{p+k-j}\gamma_{p+k+j} \; . \tag{2.8}$$

Definition 40 ([43])**.** Let $\psi(x) = \sum_{k=0}^{\infty} a_k x^k$ be a real entire function. For non-negative integers p, define the non-linear operators $\Lambda^p(z^2+1) : a_k \mapsto \Lambda_k^p(z^2+1)$, where $\Lambda_k^0(z^2+1) := a_k^2$, and for $p = 1, 2, 3, \dots$, set

$$\Lambda_k^p(z^2+1) := \binom{2p-1}{p} a_k^2 + \sum_{j=1}^{p} (-1)^j \binom{2p}{p-j} a_{k-j} a_{k+j} \; . \tag{2.9}$$

□

Example 41. Let $\psi(x) = \sum_{k=0}^{\infty} a_k x^k$ be a real entire function. Suppressing the generating polynomial z^2+1, the non-linear transformations of degree 2 and orders 1 through 7 of Definition 40 are:

$\Lambda_k^0 : a_k \mapsto a_k^2,$

$\Lambda_k^1 : a_k \mapsto a_k^2 - a_{k-1}a_{k+1},$

$\Lambda_k^2 : a_k \mapsto 3a_k^2 - 4a_{k-1}a_{k+1} + a_{k-2}a_{k+2},$

$\Lambda_k^3 : a_k \mapsto 10a_k^2 - 15a_{k-1}a_{k+1} + 6a_{k-2}a_{k+2} - a_{k-3}a_{k+3},$

$\Lambda_k^4 : a_k \mapsto 35a_k^2 - 56a_{k-1}a_{k+1} + 28a_{k-2}a_{k+2} - 8a_{k-3}a_{k+3} + a_{k-4}a_{k+4},$

$\Lambda_k^5 : a_k \mapsto 126a_k^2 - 210a_{k-1}a_{k+1} + 120a_{k-2}a_{k+2} - 45a_{k-3}a_{k+3} + 10a_{k-4}a_{k+4} - a_{k-5}a_{k+5},$

$\Lambda_k^6 : a_k \mapsto 462a_k^2 - 792a_{k-1}a_{k+1} + 495a_{k-2}a_{k+2} - 220a_{k-3}a_{k+3} + 66a_{k-4}a_{k+4} - 12a_{k-5}a_{k+5} + a_{k-6}a_{k+6}.$

□

As a joint corollary of Theorems 37 and 38, we show that the class of non-linear operators $\Lambda^p(z^2+1)$ preserves $\mathscr{L}$-$\mathscr{P}^+$.

Theorem 42 ([43])**.** *For a fixed positive integer p, let $\Lambda^p(z^2+1)$ be the non-linear operator $a_k \mapsto \Lambda_k^p(z^2+1)$ and let $\varphi(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k = \sum_{k=0}^{\infty} a_k z^k \in \mathscr{L}$-$\mathscr{P}^+$. Then, $\Lambda^p(z^2+1)[\varphi(z)] \in \mathscr{L}$-$\mathscr{P}^+$.* □

The proof follows shortly after we establish preliminary results necessary to take advantage of the equivalences in Theorems 37 and 38, from which Theorem 42 will follow. We begin by considering the class of polynomials upon which this equivalence depends and show that it belongs to $\mathscr{L}$-$\mathscr{P}^+$.

Definition 43 ([81, p. 73])**.** Given real $\alpha_1, \alpha_2, \ldots, \alpha_p$, and non-negative real $\beta_1, \beta_2, \ldots, \beta_q$, define the *generalized hypergeometric function* by

$$ {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) = 1 + \sum_{k=1}^{\infty} \frac{\prod_{n=1}^{p} (\alpha_n)_k}{\prod_{j=1}^{q} (\beta_j)_k} \cdot \frac{z^k}{k!} \ , \tag{2.10} $$

where $(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, k = 1, 2, 3, \ldots$, and $(a)_0 = 1, a \neq 0$, is the Pochhammer symbol ([81, §18]) or ascending factorial and Γ is the gamma function. □

For various properties of the Pochhammer symbol, the gamma function, and the generalized hypergeometric function, such as convergence, we refer to [81]. In the sequel we will restrict our attention mainly to hypergeometric functions of type ${}_2F_1$, which are known in the literature as *Gauss hypergeometric polynomials*, or to other ${}_pF_q$ with small p and q. Moreover, in every case we will consider at least one of the parameters appearing in the numerator of (2.10) will be a negative integer, forcing the series in the right member of (2.10) to terminate naturally and reduce to a polynomial. Hence, the hypergeometric functions to be encountered in the sequel will be referred to as *hypergeometric polynomials.*

Lemma 44 ([43])**.** *Let p be a fixed positive integer. Then, for any positive integer n,*

$$ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{S(p,k)}{2} \binom{n}{2k} z^k = \binom{2p-1}{p} \, {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; p+1; 4z \right) \ , \tag{2.11} $$

where

$$ S(p,k) = \frac{\binom{2p}{p}\binom{2k}{k}}{\binom{p+k}{p}} \ . \tag{2.12} $$

Proof. Using the fact ([81, Lemma 5, p. 22])

$$ (\alpha)_{2k} = 2^{2k} \left(\frac{\alpha}{2}\right)_k \left(\frac{1+\alpha}{2}\right)_k \ , \tag{2.13} $$

and Definition 2.10 we rewrite the right member of (2.11):

$$\begin{aligned}
\binom{2p-1}{p}\sum_{k=0}^{\infty}\frac{\left(-\frac{n}{2}\right)_k\left(\frac{1-n}{2}\right)_k}{k!(p+1)_k}(4z)^k &= \frac{1}{2}\binom{2p}{p}\sum_{k=0}^{\infty}\frac{(-n)_{2k}}{2^{2k}k!(p+1)_k}(4z)^k \\
&= \frac{1}{2}\binom{2p}{p}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-n)_{2k}}{k!}\cdot\frac{1}{(p+1)_k}z^k \\
&= \frac{1}{2}\binom{2p}{p}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n!}{k!(n-2k)!}\cdot\frac{p!}{(p+k)!}z^k \qquad (2.14)\\
&= \frac{1}{2}\binom{2p}{p}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{2k}{k}\binom{n}{2k}\cdot\frac{p!k!}{(p+k)!}z^k \\
&= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{S(p,k)}{2}\binom{n}{2k}z^k\,.
\end{aligned}$$

□

Remark 45. The location and distribution of zeros of Gauss hypergeometric polynomials has been investigated by K. Driver and K. Jordaan in [34] and based on these investigations the result of Lemma 46 appears to be a new.

Lemma 46 ([43])**.** *For fixed positive integers n and p, the zeros of the polynomial*

$$Q_n^p(z) := \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{S(p,k)}{2}\binom{n}{2k}z^k\ , \ \textit{where}\ S(p,k) = \frac{\binom{2p}{p}\binom{2k}{k}}{\binom{p+k}{p}}\ , \qquad (2.15)$$

are all real and negative.

Proof. Applying Lemma 44, we obtain

$$Q_n^p(z) = \binom{2p-1}{p}\ {}_2F_1\left(-\frac{n}{2},\frac{1-n}{2};p+1;4z\right)\ . \qquad (2.16)$$

We recall a formula relating the hypergeometric function ${}_2F_1$ and the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ ([81, formula (2), p.254]):

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\cdot\left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n,-\beta-n;\alpha+1;\frac{x-1}{x+1}\right)\ . \qquad (2.17)$$

If n is an even integer, we let $n=2m,\ m=1,2,\dots$, so that the right member of (2.16) becomes

$$\binom{2p-1}{p}\ {}_2F_1\left(-m, \frac{1-2m}{2}; p+1; 4z\right) . \tag{2.18}$$

Setting $\alpha := p, n := m, \beta := -\frac{1}{2}, x := \frac{1+4z}{1-4z}$ in (2.17), yields

$$P_m^{(p,-\frac{1}{2})}\left(\frac{1+4z}{1-4z}\right) = \frac{(1+p)_m}{m!} \cdot \left(z-\frac{1}{4}\right)^m {}_2F_1\left(-m, \frac{1-2m}{2}; p+1; 4z\right) . \tag{2.19}$$

If n is an odd integer, we let $n = 2m-1, m = 1, 2, \dots$, so that the right member of (2.16) becomes

$$\binom{2p-1}{p}\ {}_2F_1\left(1-m, \frac{1-2m}{2}; p+1; 4z\right) . \tag{2.20}$$

Setting $\alpha := p, n := m-1, \beta := \frac{1}{2}, x := \frac{1+4z}{1-4z}$ in (2.17), we obtain

$$P_{m-1}^{(p,\frac{1}{2})}\left(\frac{1+4z}{1-4z}\right) = \frac{(1+p)_{m-1}}{(m-1)!} \cdot \left(z-\frac{1}{4}\right)^{m-1} {}_2F_1\left(1-m, \frac{1-2m}{2}; p+1; 4z\right) . \tag{2.21}$$

It is well known that if $\alpha > -1$ and $\beta > -1$, then the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ are real, distinct, and lie in the interval $(-1, 1)$ (cf. [81, p. 261]). Thus, if $\gamma \in (-1, 1)$ is a zero of $P_m^{(p,-\frac{1}{2})}\left(\frac{1+4z}{1-4z}\right)$ or a zero of $P_{m-1}^{(p,\frac{1}{2})}\left(\frac{1+4z}{1-4z}\right)$, then a calculation shows that $z = \frac{\gamma-1}{4(\gamma+1)} < 0$. Therefore, the zeros of the polynomials in equations (2.19) and (2.21), and whence the zeros of the polynomial $Q_n^p(z)$, are all real and negative. $\square$

Preliminaries aside, we are now ready to prove Theorem 42.

Proof (Proof of Theorem 42)**.** Fix a positive integer p and let $\mu = \{\mu_j\}_{j=0}^{\infty}$, where $\mu_0 = \binom{2p-1}{p}, \mu_j = (-1)^j\binom{2p}{p-j}, j = 1, 2, \dots, p$. The Theorem will follow from Theorem 37 if $\varphi(z) \in \mathscr{L}\text{-}\mathscr{P}^+$ is a polynomial and if $\varphi(z)$ is transcendental, then the Theorem will follow from Theorem 38. Suppose first that $\varphi(z) \in \mathscr{L}\text{-}\mathscr{P}^+$ is a polynomial and consider the polynomial

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\left(\sum_{j=0}^{k}\frac{\mu_j}{(k+j)!(k-j)!}\right)\frac{z^k}{(n-2k)!} . \tag{2.22}$$

By part (iii) of Theorem 37, $\Lambda^p(\mu)[\varphi(z)] \in \mathscr{L}\text{-}\mathscr{P}^+$ provided we can show that the zeros of (2.22) are all real and negative. To this end, we first consider the inner sum of (2.22),

$$\frac{1}{(2k)!}\sum_{j=0}^{k}\frac{\mu_j}{(k-j)!(k+j)!} = \sum_{j=0}^{k}\binom{2k}{k-j}\mu_j$$
$$= \binom{2p-1}{p}\binom{2k}{k} + \sum_{j=1}^{k}(-1)^j\binom{2k}{k-j}\binom{2p}{p-j}. \tag{2.23}$$

We re-index the sum to absorb the term corresponding to the value of the summand when $j = 0$, and using the symmetry of the binomial coefficients, obtain

$$\frac{1}{2}\sum_{j=-k}^{k}(-1)^j\binom{2k}{k-j}\binom{2p}{p-j} = \frac{\binom{2k}{k}\binom{2p}{p}}{2\binom{p+k}{p}} = \frac{S(p,k)}{2}\ . \tag{2.24}$$

The closed form for (2.24), provided $p \geq k$, is due to K. v. Szily ([13], [85]). Thus, (2.22) becomes

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{1}{2(n-2k)!(2k)!}\cdot\frac{\binom{2k}{k}\binom{2p}{p}}{\binom{p+k}{p}}z^k = \frac{1}{n!}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{S(p,k)}{2}\binom{n}{2k}z^k, \tag{2.25}$$

and an application of Lemma 46, for each positive integer p and any positive integer n, shows that the zeros of the polynomial (2.22) are all real and negative.

Suppose now that $\varphi(z) \in \mathscr{L}\text{-}\mathscr{P}^+$ is a transcendental entire function. By part (ii) of Theorem 38, $\Lambda^p(\mu)[\varphi(z)] \in \mathscr{L}\text{-}\mathscr{P}^+$ provided

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{\mu_j}{(k+j)!(k-j)!}\right)z^k \in \mathscr{L}\text{-}\mathscr{P}^+ \cup \{0\}. \tag{2.26}$$

Calculating as in the preceding case,

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{\mu_j}{(k+j)!(k-j)!}\right)z^k = \frac{1}{2}\sum_{k=0}^{\infty}\frac{S(p,k)}{(2k)!}z^k.$$
$$= \frac{p!}{2}\binom{2p}{p}\sum_{k=0}^{\infty}\frac{1}{(k+p)!}\frac{z^k}{k!}\ , \tag{2.27}$$

which belongs to $\mathscr{L}\text{-}\mathscr{P}^+$, because $\left\{\frac{1}{(k+p)!}\right\}_{k=0}^{\infty} = \left\{\frac{1}{\Gamma(k+p+1)}\right\}_{k=0}^{\infty}$ is, by Laguerre's Theorem (Theorem 21) a CZDS, and in particular a multiplier sequence. $\square$

Corollary 47. *For each positive integer p and each positive integer n, the sequence $\{\gamma_{p,n,k}\}_{k=0}^{\infty}$, where*

$$\gamma_{p,n,k} := \binom{2p-1}{p}\binom{n}{k}^2 + \sum_{j=0}^{p}(-1)^j\binom{2p}{p-j}\binom{n}{k-j}\binom{n}{k+j}, \tag{2.28}$$

is a multiplier sequence of the first kind.

Proof. Let the sequence μ be defined as in the proof of Theorem 42. Observe that $\Lambda^p(\mu)[(x+1)^n] = \sum_{k=0}^{n}\gamma_{p,n,k}x^k$, and we have already shown in Theorem 42 that, by part (iii) of Theorem 37, for each positive integer p and each positive integer n, $\Lambda^p(\mu)[(x+1)^n] \in \mathscr{L}\text{-}\mathscr{P}^+$. $\square$

It may be instructive to express $\gamma_{p,n,k}$ in a different way. We note that

$$\binom{2n-2k}{n}\binom{n}{k-j}\binom{n}{k+j} = \binom{n}{2k}\binom{2k}{k-j}\binom{2n-2k}{n-k-j}, \tag{2.29}$$

whence

$$\begin{aligned}\binom{2n-2k}{n}\gamma_{p,n,k} := &\binom{2n-2k}{n}\binom{2p-1}{p}\binom{n}{k}^2 \\ &+ \binom{n}{2k}\sum_{j=0}^{p}(-1)^j\binom{2p}{p-j}\binom{2k}{k-j}\binom{2n-2k}{n-k-j}.\end{aligned} \tag{2.30}$$

Considering $2\gamma_{p,n,k}$ allows us to absorb the first term into the summation, because $2\binom{2p-1}{p} = \binom{2p}{p}$, and by the symmetry of the binomial coefficients, we re-index the summation to obtain

$$2\binom{2n-2k}{n}\gamma_{p,n,k} = \binom{n}{2k}\sum_{j=-p}^{p}(-1)^j\binom{2p}{p-j}\binom{2k}{k-j}\binom{2n-2k}{n-k-j}. \tag{2.31}$$

As a partial check, a calculation verifies that

$$2\binom{2n-2k}{n}\binom{2p-1}{p}\binom{n}{k}^2 = \binom{n}{2k}\binom{2p}{p}\binom{2k}{k}\binom{2n-2k}{n-k}, \tag{2.32}$$

when $j=0$, and also

$$\gamma_{1,n,k} = \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1}. \tag{2.33}$$

Next, we express the sum

$$\sum_{j=0}^{p}(-1)^j\binom{2p}{p-j}\binom{2k}{k-j}\binom{2n-2k}{n-k-j}, \tag{2.34}$$

as a hypergeometric polynomial. With the harmless assumption that $k \geq p$, we have

$$\begin{aligned} {}_3F_2(2k-2n, -k, -p; k+1, p+1; 1) &= \sum_{j=0}^{\infty} \frac{(2k-2n)_j(-k)_j(-p)_j}{j!(p+1)_j(k+1)_j} \\ &= \sum_{j=0}^{p} \frac{(-1)^{3j}(2n-2k)!(p!)^2(k!)^2}{j!(2n-2k-j)!((p-j)!)^2((k-j)!)^2} \\ &= \frac{(p!)^2(k!)^2}{(2p)!(2k)!} \sum_{j=0}^{p} (-1)^j \binom{2p}{p-j}\binom{2k}{k-j}\binom{2n-2k}{j} \\ &= \frac{1}{\binom{2p}{p}\binom{2k}{k}} \sum_{j=0}^{p} (-1)^j \binom{2p}{p-j}\binom{2k}{k-j}\binom{2n-2k}{n-k-j}, \end{aligned} \tag{2.35}$$

and together these calculations yield

$$\binom{2p}{p}\binom{2k}{k}\binom{n}{2k} {}_3F_2(2k-2n, -k, -p; k+1, p+1; 1) = 2\binom{2n-2k}{n}\gamma_{p,n,k}. \tag{2.36}$$

In Section 4.2 we will continue to explore the relationship between the zeros of hypergeometric polynomials and non-linear coefficient-wise stability preservers.

2.2.1 The class S_r of non-linear operators

S. Fisk asked whether the class of non-linear operators $a_k \mapsto S_r := a_k^2 - a_{k-r}a_{k+r},\ r = 0, 1, 2, \ldots$, acting on functions of the form $\sum_{k=0}^{n} a_k x^k$, takes polynomials with only real negative zeros into polynomials of the same type ([36, Question 2]). As with the class Λ^p, we follow our usual conventions when denoting the non-linear operator S_r (cf. Notation 36). S_0 produces the zero polynomial and P. Brändén ([9]) confirmed the cases when $r = 1, 2, 3$. R. Yoshida confirmed the case when $r = 4$ and produced a counterexample in the case $r = 6$ ([91]). The non-linear operator S_5 is known to preserve $\mathscr{L}$-$\mathscr{P}^+$ ([92]), and the behavior of $S_r, r \geq 6$, may be studied in light of the following observation.

Proposition 48. *For any positive integer r,*

$$S_r[e^x] = I_0(2\sqrt{x}\,) - I_{2r}(2\sqrt{x}\,), \tag{2.37}$$

where $I_n(x)$ is the modified Bessel function of the first kind of index n (see [81, §58]*).* □

We show that the elements of the class S_r are a natural building block for the operators $\Lambda^p(z^2+1)$, and in particular, the non-linear operators S_r and $\Lambda^p(z^2+1)$ are related in a remarkable way.

Proposition 49 ([43])**.** *Fix a positive integer p and let $\Lambda^p(z^2+1) : a_k \mapsto \Lambda^p_k(z^2+1)$ be the non-linear operator of Definition 40. Then,*

$$\Lambda^p_k(z^2+1) = \sum_{j=1}^{p} (-1)^{j+1} \binom{2p}{p-j} (a_k^2 - a_{k-j}a_{k+j}) \ . \tag{2.38}$$

Proof. For each positive integer p we have the identity

$$0 = \sum_{j=-p}^{p} (-1)^j \binom{2p}{p-j} = \binom{2p}{p} + 2\sum_{j=1}^{p} (-1)^j \binom{2p}{p-j} \ , \tag{2.39}$$

which justifies the following calculation. Using Definition 2.9 and (2.39),

$$\begin{aligned}\Lambda^p_k(z^2+1) &= a_k^2 \sum_{j=1}^{p} (-1)^{j+1} \binom{2p}{p-j} + \sum_{j=1}^{p} (-1)^j \binom{2p}{p-j} a_{k-j}a_{k+j} \\ &= \binom{2p-1}{p} a_k^2 + \sum_{j=1}^{p} (-1)^j \binom{2p}{p-j} a_{k-j}a_{k+j} \ .\end{aligned} \tag{2.40}$$

□

In Section 4.1.3 we will establish the concavity preserving properties of S_r. However, for reasons to be discussed later, we may not conclude from Proposition 49 that $\Lambda^p(z^2+1)$ enjoys the same properties. Another relationship between S_r and $\Lambda^p(z^2+1)$, given below, will allow us to give an integral representation of S_r and then by extension for $\Lambda^p(z^2+1)$, too.

Proposition 50 ([43])**.** *If the zeros of the real polynomial $\varphi(x) = \sum_{k=0}^{n} a_k x^k$ are all real and negative, then for any positive integer p, the zeros of the polynomial*

$$\sum_{j=1}^{p} \left((-1)^{j+1} \binom{2p}{p-j} \sum_{k=0}^{n} (a_k^2 - a_{k-j}a_{k+j}) x^k \right) \tag{2.41}$$

are all real and negative.

Proof. Suppose that the zeros of the polynomial $\varphi(x) = \sum_{k=0}^{n} a_k x^k$ are all real and negative. By Theorem 42, the zeros of the polynomial $\Lambda^p(z^2+1)\,[\varphi(x)]$ are all real and negative for any positive integer p, and by Proposition 49

$$\begin{aligned}\Lambda^p(z^2+1)\left[\varphi(x)\right] &= \sum_{k=0}^{n}\left(\sum_{j=1}^{p}(-1)^{j+1}\binom{2p}{p-j}(a_k^2-a_{k-j}a_{k+j})\right)x^k \\ &= \sum_{j=1}^{p}\left((-1)^{j+1}\binom{2p}{p-j}\sum_{k=0}^{n}(a_k^2-a_{k-j}a_{k+j})x^k\right) .\end{aligned} \tag{2.42}$$

□

2.2.2 Integral representation of the class $\Lambda^p(z^2+1)$

The non-linear operator $\Lambda^0(z^2+1) : a_k \mapsto a_k^2$ may be regarded as the Hadamard product (see §1.6.2) of $\varphi(x)$ with itself. Using a classical result of J. Hadamard, we obtain an integral representation for the class on non-linear operators S_r, and show that the class $\Lambda^p(z^2+1)$ admits a similar integral representation.

Theorem 51 ([45], [86])**.** *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(D(0,R))$ and let $g(z) = \sum_{k=0}^{\infty} b_k z^k \in H(D(0,R'))$. Then for $z \in D(0,RR')$,*

$$\sum_{k=0}^{\infty} a_k b_k z^k = \frac{1}{2\pi i}\int_\gamma f(w)g\left(\frac{z}{w}\right)\,\frac{dw}{w}\ , \tag{2.43}$$

where γ is a closed rectifiable curve containing the origin on which $|w| < R$ and $\left|\frac{z}{w}\right| < R'$.

Proof. The uniform convergence of the series permits term-by-term integration, whence

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma f(w)g\left(\frac{z}{w}\right)\,\frac{dw}{w} &= \frac{1}{2\pi i}\int_\gamma f(w)\sum_{k=0}^{\infty}\frac{b_k z^k}{w^{k+1}}\,dw \\ &= \sum_{k=0}^{\infty}\left(b_k z^k\cdot\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{k+1}}\,dw\right) \\ &= \sum_{k=0}^{\infty} a_k b_k z^k\ .\end{aligned} \tag{2.44}$$

□

Theorem 52 ([43])**.** *For each positive integer p, the non-linear operator $\Lambda^p(z^2+1)$ acting on entire functions of the form $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has the integral representation $\int_\gamma f(w)f\left(\frac{z}{w}\right)K_p(z,w)\,\frac{dw}{w}$, where $K_p(z,w) = \sum_{j=1}^{p}(-1)^{j+1}\binom{2p}{p-j}\left(1-\frac{w^{2j}}{z^j}\right)$, and γ is a closed rectifiable curve containing the origin.*

Proof. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be entire and let γ be a closed rectifiable curve containing the origin. Term-by-term integration is permitted on γ, and we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma f(z) f\left(\frac{z}{w}\right)\left(1 - \frac{w^{2j}}{z^j}\right)\frac{dw}{w} = & \frac{1}{2\pi i}\int_\gamma f(z) f\left(\frac{z}{w}\right)\frac{dw}{w} \\
& - \frac{1}{2\pi i}\int_\gamma f(w)\sum_{k=0}^{\infty}\frac{a_k z^{k-j}}{w^{k-2j+1}}\, dw \\
= & \sum_{k=0}^{\infty} a_k^2 z^k - \sum_{k=0}^{\infty}\left(a_k \cdot \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{k-2j+1}}\, dw\right) z^{k-j} \\
= & \sum_{k=0}^{\infty} a_k^2 z^k - \sum_{k=0}^{\infty} a_k a_{k-2j} z^{k-j} \qquad (2.45) \\
= & \sum_{k=0}^{\infty} a_k^2 z^k - \sum_{k=-j}^{\infty} a_{k-j} a_{k+j} z^k \\
= & \sum_{k=0}^{\infty} (a_k^2 - a_{k-j} a_{k+j}) z^k \\
= & S_j[f(z)] \, .
\end{aligned}$$

By Proposition 49, for any positive integer p, we have the integral representation

$$\begin{aligned}
\Lambda^p(z^2+1)[f(z)] &= \sum_{k=0}^{\infty}\sum_{j=1}^{p}(-1)^{j+1}\binom{2p}{p-j}(a_k^2 - a_{k-j}a_{k+j}) z^k \\
&= \sum_{j=1}^{p}(-1)^{j+1}\binom{2p}{p-j}\left(\frac{1}{2\pi i}\int_\gamma f(z) f\left(\frac{z}{w}\right)\left(1 - \frac{w^{2j}}{z^j}\right)\frac{dw}{w}\right) \qquad (2.46) \\
&= \frac{1}{2\pi i}\int_\gamma f(z) f\left(\frac{z}{w}\right)\left(\sum_{j=1}^{p}(-1)^{j+1}\binom{2p}{p-j}\left(1 - \frac{w^{2j}}{z^j}\right)\right)\frac{dw}{w} \, .
\end{aligned}$$

□

CHAPTER 3
HIGHER ORDER NON-LINEAR OPERATORS

3.1 Arithmetic of non-linear operators of type $\Lambda^p(\mu)$.

3.1.1 A characterization of $\Lambda^p(z^2+1)$

The aim of this section is to further classify the sequences of real numbers $\mu = \{\mu_j\}_{j=0}^{\infty}$ for which the non-linear operator $\Lambda(\mu)$ (cf. Theorem 38) preserves $\mathscr{L}$-$\mathscr{P}^+$. Thus far we know that $\{1, -1, 0, 0, \ldots\}$ is an example of such a sequence (see Brändén's Theorem or [9, Conjecture 1.1]) and Theorem 42 states that setting $\mu_0 := \binom{2p-1}{p}$, $\mu_j := (-1)^j \binom{2p}{p-j}$, $j = 1, 2, \ldots, p$, for each positive integer p, produces a class of such sequences. The characterization of the non-linear operators given in [9, Theorems 4.2 and 5.7] (Theorems 37 and 38) follows in the spirit of the Pólya-Schur algebraic and transcendental characterization of classical multiplier sequences (see §1.4.1) and we follow a similar approach. We give the explicit range of values for μ_0 and μ_2 for which the non-linear operators $\Lambda^p(\{\mu_0, \mu_2, 0, 0\ldots\})$ preserve $\mathscr{L}$-$\mathscr{P}^+$ and show how our technique may be used to obtain explicit bounds on the elements of the generating sequence μ for which the associated non-linear operators preserves $\mathscr{L}$-$\mathscr{P}^+$.

Theorem 53. *Let $\mu = \{\mu_{2j}\}_{j=0}^{p}$ be a real sequence. For each positive integer p the non-linear operator $\Lambda^p : a_k \mapsto \Lambda_k^p(\mu)$ preserves $\mathscr{L}$-$\mathscr{P}^+$, provided that the zeros of either one (or both) the polynomials $h_p(x)$ and $\tilde{h}_p(x)$, where*

$$\begin{aligned} h_p(k) &= p!\binom{p+k}{k}\sum_{j=0}^{p}(-1)^j\frac{(-k)_j}{(k+1)_j}\mu_{2j}, \ \text{and} \\ e^x\tilde{h}_p(x) &= \sum_{k=0}^{\infty}\frac{h_p(k)}{k!}x^k, \end{aligned} \tag{3.1}$$

are all real and negative.

Proof. Consider the expressions in (iii) of Theorem 37 and (ii) of Theorem 38,

$$\begin{aligned}\sum_{j=0}^{p}\left(\frac{\mu_{2j}}{(k-j)!(k+j)!}\right) &= \frac{1}{(2k)!}\sum_{j=0}^{p}\binom{2k}{k-j}\mu_{2j} \\ &= \frac{1}{k!k!}\sum_{j=0}^{p}\frac{\prod_{n=1}^{j}(k-n+1)}{\prod_{n=1}^{j}(k+n)}\mu_{2j} \\ &= \frac{1}{k!k!}\sum_{j=0}^{p}(-1)^j\frac{(-k)_j}{(k+1)_j}\mu_{2j},\end{aligned} \tag{3.2}$$

which up to a constant factor simplifies to a polynomial of degree p in the variable k,

$$h_p(k) := p!\binom{p+k}{k}\sum_{j=0}^{p}(-1)^j\frac{(-k)_j}{(k+1)_j}\mu_{2j}. \tag{3.3}$$

An application of Lemma 46 (cf. Theorem 42) establishes the equivalences in Theorems 37 and 38. Indeed,

$$\begin{aligned}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\left(\frac{1}{k!k!}\sum_{j=0}^{p}(-1)^j\frac{(-k)_j}{(k+1)_j}\mu_{2j}\right)\frac{z^k}{(n-2k)!} &= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{h_p(k)}{\binom{p+k}{k}}\frac{z^k}{p!k!k!(n-2k)!} \\ &= \frac{1}{n!}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\binom{2k}{k}\frac{h_p(k)}{\binom{p+k}{k}}\frac{z^k}{p!} \\ &= \frac{p!}{(2p)!n!}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\binom{2p}{p}\binom{2k}{k}\frac{h_p(k)}{\binom{p+k}{k}}z^k \\ &= \frac{p!}{(2p)!n!}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}S(p,k)\binom{n}{2k}h_p(k)z^k\ .\end{aligned} \tag{3.4}$$

□

For each positive integer p, we consider the expressions in (iii) of Theorem 37 and (ii) of Theorem 38, and ask that for each sequence $\mu = \{\mu_{2j}\}_{j=0}^{p}$, the expressions each interpolate a CZDS and a multiplier sequence of the first kind. Unfortunately, computations of this kind quickly become too difficult, and even with the aid of a computer the problem of expressing the coefficients of the polynomial $h_p(k)$ seems intractable in all but a few special cases.

We remark also that the transformation $h_p(x) \mapsto \tilde{h}_p(x)$ is obtained by means of the transformation of basis $x^n \mapsto B_n(x), n = 1, 2, 3, \ldots$, where $B_n(x)$ denotes the Bell polynomial of degree n (see §4.1.4). We will take up questions related to the stability of Bell polynomials in Section 4.1.4.

Example 54. The first several of the polynomials $h_p(x)$ are

$$
\begin{aligned}
h_1(x) &= \mu_0 + (\mu_0 + \mu_2)x, \\
h_2(x) &= 2\mu_0 + (3\mu_0 + 2\mu_2 - \mu_4)x + (\mu_0 + \mu_2 + \mu_4)x^2, \\
h_3(x) &= 6\mu_0 + (11\mu_0 + 6\mu_2 - 3\mu_4 + 2\mu_6)x + (6\mu_0 + 5\mu_2 + 2\mu_4 - 3\mu_6)x^2 \\
&\qquad + (\mu_0 + \mu_2 + \mu_4 + \mu_6)x^3, \\
h_4(x) &= 24\mu_0 + (50\mu_0 + 24\mu_2 - 12\mu_4 + 8\mu_6 - 6\mu_8)x \\
&\qquad + (35\mu_0 + 26\mu_2 + 5\mu_4 - 10\mu_6 + 11\mu_8)x^2 + (10\mu_0 + 9\mu_2 + 6\mu_4 + \mu_6 - 6\mu_8)x^3 \\
&\qquad + (\mu_0 + \mu_2 + \mu_4 + \mu_6 + \mu_8)x^4,
\end{aligned}
\tag{3.5}
$$

and the first several of the polynomials $\tilde{h}_p(x)$ generated by $h_p(x)$ by means of the transformation $e^x\tilde{h}_p(k) = \sum_{k=0}^{\infty} \frac{h_p(k)}{k!}x^k$, are

$$
\begin{aligned}
\tilde{h}_1(x) &= \mu_0 + (\mu_0 + \mu_2)x, \\
\tilde{h}_2(x) &= 2\mu_0 + (4\mu_0 + 3\mu_2)x + (\mu_0 + \mu_2 + \mu_4)x^2, \\
\tilde{h}_3(x) &= 6\mu_0 + (18\mu_0 + 12\mu_2)x + (9\mu_0 + 8\mu_2 + 5\mu_4)x^2 + (\mu_0 + \mu_2 + \mu_4 + \mu_6)x^3, \\
\tilde{h}_4(x) &= 24\mu_0 + (96\mu_0 + 60\mu_2)x + (72\mu_0 + 60\mu_2 + 30\mu_4)x^2 + (16\mu_0 + 15\mu_2 + 12\mu_4 + 7\mu_6)x^3 \\
&\qquad + (\mu_0 + \mu_2 + \mu_4 + \mu_6 + \mu_8)x^4.
\end{aligned}
\tag{3.6}
$$

As a corollary, we explicitly state the sufficient conditions on the coefficient sequences $\{\mu_0, \mu_2, 0, 0, \ldots\}$ for which the non-linear operator $a_k \mapsto \mu_0 a_k^2 + \mu_2 a_{k-1}a_{k+2}$ preserves $\mathscr{L}$-$\mathscr{P}^+$. Already, the next case yields several dozen inequalities which characterize the coefficient sequences $\{\mu_0, \mu_2, \mu_4, 0, 0, \ldots\}$ that preserve $\mathscr{L}$-$\mathscr{P}^+$ via the non-linear operator $\Lambda^2(\{\mu_0, \mu_2, \mu_4, 0, 0, \ldots\})$, and the general case appears intractable. We will remedy this problem in the next section and obtain an alternative means of characterizing the coefficient sequences μ for which the non-linear operators $a_k \mapsto \Lambda_k^p(\mu)$ preserve $\mathscr{L}$-$\mathscr{P}^+$.

Corollary 55. *The non-linear operator $a_k \mapsto \mu_0 a_k^2 + \mu_2 a_{k-1}a_{k+1}, \mu_0 \neq -\mu_2$, acting on real entire functions of the form $\sum_{k=0}^{\omega} a_k z^k, 0 \leq \omega \leq \infty$, preserves $\mathscr{L}$-$\mathscr{P}^+$ provided*

(i) $\mu_2 \neq 0, \mu_0 > 0$, *and* $\mu_0 + \mu_2 > 0$, *or*

(ii) $\mu_2 \neq 0, \mu_0 < 0$, *and* $\mu_0 + \mu_2 < 0$, *or*

(iii) exactly one of μ_0, μ_2 *is zero.*

Proof. Suppose first that the non-linear operator $a_k \mapsto \mu_0 a_k^2 + \mu_2 a_{k-1}a_{k+1}$, is acting on polynomials in $\mathscr{L}$-$\mathscr{P}^+$. By part (iii) of Theorem 37, we have to show that the zeros of the polynomial

$$\begin{aligned}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{\mu_0}{k!k!} + \frac{\mu_2}{(k-1)!(k+1)!} \right) \frac{z^k}{(n-2k)!} &= \frac{1}{n!} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2k}{k} \binom{n}{2k} \left(\frac{k(\mu_0+\mu_2)+\mu_0}{k+1} \right) z^k \\ &= \frac{1}{n!} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{S(1,k)}{2} (k(\mu_0+\mu_2)+\mu_2) z^k,\end{aligned} \tag{3.7}$$

are all real and negative. We have already shown in Lemma 46 that, for each positive integer p and all positive integers n, the zeros of the polynomials

$$Q_n^p(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{S(p,k)}{2} \binom{n}{2k} z^k \text{, where } S(p,k) := \frac{\binom{2p}{p}\binom{2k}{k}}{\binom{p+k}{k}}, \tag{3.8}$$

are all real and negative, hence we may assume that $\mu_0 \neq -\mu_2$. To complete the proof it suffices to show that the zeros of the polynomial (3.7) are all real and negative for each positive integer n. To this end, we view $\{k(\mu_0+\mu_2)+\mu_0\}_{k=0}^{\infty}$ as a possible CZDS or a possible multiplier sequence of the first kind, and let $h_1(x) = x(\mu_0+\mu_2)+\mu_0$ be the interpolating polynomial.

Suppose that $\{h_1(k)\}_{k=0}^{\infty}$ interpolates a CZDS. If case (i) or case (ii) holds, then $h_1(0) \neq 0$ and all the zeros of $h_1(x)$ are real and negative, whence $\{h_1(k)\}_{k=0}^{\infty}$ is a CZDS by part (i) of Theorem 22. If instead case (iii) holds, then none of the zeros of $h_1(x)$ lie in the interval $(0,1)$ and $\{h_1(k)\}_{k=0}^{\infty}$ is a CZDS by (i) of Theorem 21. By Remark 23, assuming that $h_1(x)$ interpolates a CZDS may not yield the complete range of values, so suppose next that $h_1(x)$ interpolates a multiplier sequence of the first kind. Calculating as in (ii) of Theorem 17, $h_1(x)e^x = \sum_{k=0}^{\infty} \frac{h_1(k)}{k!} x^k$, and under the hypotheses of the Theorem the zeros of $h_1(x)$ are real and negative.

Finally, let the non-linear operator act on a transcendental function $\sum_{k=0}^{\infty} a_k z^k \in \mathscr{L}$-$\mathscr{P}^+$. Using part (ii) of Theorem 38,

$$\sum_{k=0}^{\infty}\left(\frac{\mu_0}{k!k!}+\frac{\mu_2}{(k-1)!(k+1)!}\right)z^k=\sum_{k=0}^{\infty}\left(\frac{k(\mu_0+\mu_2)+\mu_0}{(k+1)!}\right)\frac{z^k}{k!}, \tag{3.9}$$

which belongs to $\mathscr{L}$-$\mathscr{P}^+$, because the sequence $\left\{\frac{1}{(k+1)!}\right\}_{k=0}^{\infty}=\left\{\frac{1}{\Gamma(k)}\right\}_{k=0}^{\infty}$ is by Laguerre's Theorem (Theorem 21) a CZDS. □

We already remarked that setting $\mu_0=-\mu_2$ reduces Corollary 55 to Theorem 35. Similarly, the choice of $\mu_0=3,\mu_2=-4,\mu_4=1$ produces $h_2(x)=\tilde{h}_2(x)=6$, whence the non-linear operator $\Lambda^2(z^2+1)$ satisfies Theorem 53 trivially. This appears to be the case in general: setting $\{\mu_{2j}\}_{j=0}^{p}$, where $\mu_0=\binom{2p-1}{p},\mu_{2j}=(-1)^j\binom{2p}{p-j}, j=1,2,\ldots,p$, in (3.3) produces a constant. However, repeating the calculation with the sequence $\{\mu_j\}_{j=0}^{p}$, where $\mu_0=\binom{2p-1}{p},\mu_{2j}=\binom{2p}{p-j}, j=1,2,\ldots,p$, where all terms are positive integers, leads to a particularly pleasant class of polynomials: their zeros are all located at the negative half-integers (cf. Sloane's A161198).

In contrast to the non-linear operator $\Lambda^1(z^2+1): a_k\mapsto a_k^2-a_{k-1}a_{k+1}$, letting $\mu_0=1,\mu_2=-1,\mu_4=1$, we see that $h_2(x)=x^2+2$ and $\tilde{h}_2(x)=x^2+x+2$ and we expect that the the non-linear operator $a_k\mapsto a_k^2-a_{k-1}a_{k+1}+a_{k-2}a_{k+2}$ does not preserve $\mathscr{L}$-$\mathscr{P}^+$. Indeed, it does not preserve $\mathscr{L}$-$\mathscr{P}$ as can be see by considering the polynomial $(x+1)^2(x-2)$, and to see that it also does not preserve $\mathscr{L}$-$\mathscr{P}^+$ one must consider polynomials of higher degree, for example $(1+x)^9$.

We apply Theorem 53 to the S_r class on non-linear operators. For $r\geq 2$ we may write $S_r=\Lambda^r(\{1,0,\ldots,0,1\})$, where the 1's are in the first and r^{th} position, and we have the polynomials

$$\begin{aligned}
h_2(x)&=2(x^2+x+1),\\
h_3(x)&=2x^3+3x^2+13x+6,\\
h_4(x)&=2(x^4+2x^3+23x^2+22k+12),\\
h_5(x)&=2x^5+5x^4+120k^3+175x^2+298x+120,
\end{aligned} \tag{3.10}$$

and

$$
\begin{aligned}
\tilde{h}_2(x) &= 2(1+x)^2, \\
\tilde{h}_3(x) &= 2x^3 + 9x^2 + 18x + 6, \\
\tilde{h}_4(x) &= 2(x^4 + 8x^3 + 36x^2 + 48x + 12), \\
\tilde{h}_5(x) &= 2x^5 + 25x^4 + 200x^3 + 600x^2 + 600x + 120.
\end{aligned} \tag{3.11}
$$

It can be checked with the aid of a computer that, apart from $\tilde{h}_2(x)$, all of these polynomials have non-real zeros, hence the operators $S_r, r = 3, 4, 5$, fail to satisfy the conclusion of Theorem 53 even though for $r = 2, 3, 4, 5$, the non-linear operator S_r is known to preserve $\mathscr{L}\text{-}\mathscr{P}^+$.

We conclude this section by stating the sufficient conditions under which the non-linear operator $\Lambda^2(z^2+1) : a_k \mapsto \mu_0 a_k^2 + \mu_2 a_{k-1}a_{k+1} + \mu_4 a_{k-2}a_{k+2}$ preserves $\mathscr{L}\text{-}\mathscr{P}^+$. The inequalities listed below have been obtained with the aid of Mathematica.

Corollary 56. *The non-linear operator* $\Lambda^2(z^2+1) : a_k \mapsto \mu_0 a_k^2 + \mu_2 a_{k-1}a_{k+1} + \mu_4 a_{k-2}a_{k+2}$ *preserves* $\mathscr{L}\text{-}\mathscr{P}^+$*, provided the zeros of either (or both) of the polynomials*

$$
\begin{aligned}
h_2(x) &= 2\mu_0 + (3\mu_0 + 2\mu_2 - \mu_4)x + (\mu_0 + \mu_2 + \mu_4)x^2, \text{ and} \\
\tilde{h}_2(x) &= 2\mu_0 + (4\mu_0 + 3\mu_2)x + (\mu_0 + \mu_2 + \mu_4)x^2,
\end{aligned} \tag{3.12}
$$

are all real an negative. Explicitly, the zeros of $h_2(x)$ *and* $\tilde{h}_2(x)$ *are real and negative provided the inequalities listed below hold, where the inequalities marked with (*) guarantee that the zeros of* $\tilde{h}_2(x)$ *are all real and negative.*

(i) $\mu_2 < 0$ *and*

- *(a)* $\mu_4 \leq \mu_2$ *and* $\mu_0 \leq 2\mu_2 + 7\mu_4 - 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$*, or*
- *(b)* $\mu_4 \leq \mu_2$ *and* $\mu_0 > \mu_2 - \mu_4$*, or*
- *(c)* $\mu_2 < \mu_4 \leq \frac{\mu_2}{4}$ *and* $\mu_0 \leq 2\mu_2 + 7\mu_4 - 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$*, or*
- *(d)* $\mu_2 < \mu_4 \leq \frac{\mu_2}{4}$ *and* $\mu_0 \geq 0$*, or*
- *(e)* $\frac{\mu_2}{4} < \mu_4 < 0$ *and* $\mu_0 \leq 2\mu_2 + 7\mu_4 - 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$*, or*
- *(f)* $\frac{\mu_2}{4} < \mu_4 < 0$ *and* $2\mu_2 + 7\mu_4 + 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)} \leq \mu_0 < \mu_2 - \mu_4$*, or*
- *(g)* $\frac{\mu_2}{4} < \mu_4 < 0$ *and* $\mu_0 \geq 0$*, or*

(h) $0 \le \mu_4 \le -\frac{\mu_2}{2}$ *and* $\mu_0 < \mu_2 - \mu_4$, *or*

(i) $0 \le \mu_4 \le -\frac{\mu_2}{2}$ *and* $\mu_0 \ge 0$, *or*

(j) $-\frac{\mu_2}{2} < \mu_4 < -2\mu_2$ *and* $\mu_0 < \mu_2 - \mu_4$, *or*

(k) $-\frac{\mu_2}{2} < \mu_4 < -2\mu_2$ *and* $0 \le \mu_0 \le 2\mu_2 + 7\mu_4 - 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$, *or*

(l) $-\frac{\mu_2}{2} < \mu_4 < -2\mu_2$ *and* $\mu_0 \ge 2\mu_2 + 7\mu_4 + 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$, *or*

(m) (*) $\mu_0 < \dfrac{3\mu_2}{4}$ *and* $\dfrac{8\mu_0^2 - 16\mu_0\mu_2 + 9\mu_2^2}{8\mu_0} \le \mu_4 < \mu_2 - \mu_0$, *or*

(n) (*) μ_0 *and* $\mu_4 > \mu_2$, *or*

(o) (*) $\mu_0 > 0$ *and* $\mu_2 - \mu_0 < \mu_4 \le \dfrac{8\mu_0^2 - 16\mu_0\mu_2 + 9\mu_2^2}{8\mu_0}$, *or*

(ii) $\mu_4 = -2\mu_2$ *and*

(a) $\mu_0 < 3\mu_2$, *or*

(b) $\mu_0 = 0$, *or*

(c) $\mu_0 \ge 12 - \mu_2 + \sqrt{\mu_2^2}$, *or*

(iii) $\mu_4 > -2\mu_2$ *and*

(a) $\mu_0 < \mu_2 - \mu_4$, *or*

(b) $\mu_0 \ge 2\mu_2 + 7\mu_4 + 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$, *or*

(iv) $\mu_2 = 0$ *and*

(a) $\mu_4 < 0$ *and* $\mu_0 \le 7\mu_4 - 4\sqrt{3}\sqrt{\mu_4^2}$, *or*

(b) $\mu_4 < 0$ *and* $\mu_0 > -\mu_4$, *or*

(c) $\mu_4 = 0$ *and* $\mu_0 < 0$, *or*

(d) $\mu_4 = 0$ *and* $\mu_0 > 0$, *or*

(e) $\mu_4 > 0$ *and* $\mu_0 < -\mu_4$, *or*

(f) $\mu_4 > 0$ *and* $\mu_0 \ge 7\mu_4 + 4\sqrt{3}\sqrt{\mu_4^2}$, *or*

(g) (*) $\mu_0 < 0$ *and* $\mu_0 \le \mu_4 < -\mu_0$, *or*

(h) (*) $\mu_0 = 0$ *and* $\mu_4 \ne 0$, *or*

(i) (*) $\mu_0 > 0$ *and* $-\mu_0 < \mu_4 \le \mu_0 <$, *or* *(v)* $\mu_2 > 0$ *and*

(a) $\mu_4 < -2\mu_2$ *and* $\mu_0 \le 2\mu_2 + 7\mu_4 - 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$*, or*

(b) $\mu_4 < -2\mu_2$ *and* $\mu_0 > \mu_2 - \mu_4$*, or*

(c) $\mu_4 = -2\mu_2$ *and* $\mu_0 \le -12\mu_2 + \sqrt{\mu_2^2}$*, or*

(d) $\mu_4 = -2\mu_2$ *and* $\mu_0 = 0$*, or*

(e) $\mu_4 = -2\mu_2$ *and* $\mu_0 > 3\mu_2$*, or*

(f) $-2\mu_2 < \mu_4 < -\frac{\mu_2}{2}$ *and* $\mu_0 \le 2\mu_2 + 7\mu_4 - 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$*, or*

(g) $-2\mu_2 < \mu_4 < -\frac{\mu_2}{2}$ *and* $2\mu_2 + 7\mu_4 + 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)} \le \mu_0 \le 0$*, or*

(h) $-2\mu_2 < \mu_4 < -\frac{\mu_2}{2}$ *and* $\mu_0 > \mu_2 - \mu_4$*, or*

(i) $-\frac{\mu_2}{2} \le \mu_4 \le 0$ *and* $\mu_0 \le 0$*, or*

(j) $-\frac{\mu_2}{2} \le \mu_4 \le 0$ *and* $\mu_0 > \mu_2 - \mu_4$*, or*

(k) $0 < \mu_4 < \frac{\mu_2}{4}$ *and* $\mu_0 \le 0$*, or*

(l) $0 < \mu_4 < \frac{\mu_2}{4}$ *and* $\mu_2 - \mu_4 < \mu_0 \le 2\mu_2 + 7\mu_4 - 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$*, or*

(m) $0 < \mu_4 < \frac{\mu_2}{4}$ *and* $\mu_0 \ge 2\mu_2 + 7\mu_4 + 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$*, or*

(n) $\frac{\mu_2}{4} \le \mu_4 < \mu_2$ *and* $\mu_0 \le 0$*, or*

(o) $\frac{\mu_2}{4} \le \mu_4 < \mu_2$ *and* $\mu_0 \ge 2\mu_2 + 7\mu_4 + 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$*, or*

(p) $\mu_4 = \mu_2$ *and* $\mu_0 < 0$*, or*

(q) $\mu_4 = \mu_2$ *and* $\mu_0 \ge 33\mu_2 + 2\sqrt{2}\sqrt{\mu_2^2}$*, or*

(r) $\mu_4 > \mu_2$ *and* $\mu_0 < \mu_2 - \mu_4$*, or*

(s) $\mu_4 > \mu_2$ *and* $\mu_0 \ge 2\mu_2 + 7\mu_4 + 2\sqrt{6}\sqrt{\mu_4(\mu_2 + 2\mu_4)}$*, or*

(t) *(*)* $\mu_0 < 0$ *and* $\dfrac{8\mu_0^2 - 16\mu_0\mu_2 + 9\mu_2^2}{8\mu_0} \le \mu_4 < \mu_2 - \mu_0$*, or*

(u) *(*)* $\mu_0 = 0$ *and* $\mu_4 < \mu_2$*, or*

(v) *(*)* $\mu_0 > \dfrac{3\mu_2}{4}$ *and* $\mu_2 - \mu_0 < \mu_4 \le \dfrac{8\mu_0^2 - 16\mu_0\mu_2 + 9\mu_2^2}{8\mu_0}$.

□

3.1.2 Arithmetic of the generating polynomials

Theorem 55 gives an explicit characterization of the non-linear operators $a_k \mapsto \Lambda^1_k(z^2+1)$ that preserve $\mathscr{L}$-$\mathscr{P}^+$, but does not lend itself to the algebraic properties of $\Lambda^1(z^2+1)$. Consider the following example.

Example 57. A calculation using the Cardon expressions (1.7) shows that $\Lambda^1(2-\sqrt{2}x+x^2) = \Lambda^1(\{2,-1,0,0,\ldots\})$ and $\Lambda^1(-1-2x+x^2) = \Lambda^1(\{-1,3,0,0,\ldots\})$. According to Theorem 59, the former preserves $\mathscr{L}$-$\mathscr{P}^+$ while the latter does not. Indeed, the latter operator is generated by a polynomial with two distinct real zeros. Adding the generating sequences term-wise we obtain the non-linear operator $a_k \mapsto a_k^2 + 2a_{k-1}a_{k+1}$, which does satisfy the hypotheses of Theorem 55. □

Example 58. According to Theorem 59, the non-linear operators $a_k \mapsto 3a_k^2 - 2a_{k-1}a_{k+1}$ and $a_k \mapsto -2a_k^2 - a_{k-1}a_{k+1}$ both preserve $\mathscr{L}$-$\mathscr{P}^+$. The non-linear operator, $a_k \mapsto a_k^2 - 3a_{k-1}a_{k+1}$, however, does not preserve $\mathscr{L}$-$\mathscr{P}^+$ as can be see by considering the polynomial $(x+1)^2$. □

The above example suggest that the usual approach to adding the generating sequences will not preserve the properties of the non-linear operators we have been considering. We wish to remedy this situation and define the usual arithmetic operations on sequences that make sense in this setting and that preserve the properties of the associated non-linear operators. For each positive integer p the sequences $\mu = \{\mu_j\}_{j=0}^{\infty}$ for which the operator $\Lambda^p(\mu)$ preserves $\mathscr{L}$-$\mathscr{P}^+$ were described in terms of certain interpolating polynomials. We now give another description of such sequences.

Let $a+bi$ and $a-bi$ be the zeros of the even polynomial $g(z)$ in Theorem 8 and let $f(z)$ be an entire function. We evaluate the expression

$$A^p_k(z) = \frac{1}{k!}\frac{d^k}{dt^k}\left[f^{(p)}(z+(a+bi)t)f^{(p)}(z+(a-bi)t)\right]\Big|_{t=0}, \tag{3.13}$$

for several values of k and obtain:

$$
\begin{aligned}
\frac{2!}{2}A_2^p(z) &= (a^2+b^2)(f^{(p+1)}(z))^2 + (a^2-b^2)f^{(p)}(z)f^{(p+2)}(z)\ , \\
\frac{4!}{2}A_4^p(z) &= 3(a^2+b^2)^2(f^{(p+2)}(z))^2 + 4(a^4-b^4)f^{(p+1)}(z)f^{(p+3)}(z) \\
&\qquad + (a^4-6a^2b^2+b^4)f^{(p)}(z)f^{(p+4)}(z)\ , \\
\frac{6!}{2}A_6^p(z) &= 10(a^2+b^2)^3(f^{(p+3)}(z))^2 + 15(a^2+b^2)^2(a^2-b^2)f^{(p+2)}(z)f^{(p+4)}(z) \\
&\qquad + 6(a^6-5a^4b^2-5a^2b^4+b^6)f^{(p+1)}(z)f^{(p+5)}(z) \\
&\qquad + (a^6-15a^4b^2+15a^2b^4-b^6)f^{(p)}(z)f^{(p+6)}(z)\ , \\
\frac{8!}{2}A_8^p(z) &= 35(a^2+b^2)^4(f^{(p+4)}(z))^2 + 56(a^2+b^2)^2(a^4-b^4)f^{(p+3)}(z)f^{(p+5)}(z) \\
&\qquad + 28(a^2+b^2)^2(a^6-5a^4b^2-5a^2b^4+b^6)f^{(p+2)}(z)f^{(p+6)}(z) \\
&\qquad + 8(a^2+b^2)(a^6-15a^4b^2+15a^2b^4-b^6)f^{(p+1)}(z)f^{(p+7)}(z) \\
&\qquad + (a^8-28a^6b^2+70a^4b^4-28a^2b^6+b^8)f^{(p)}(z)f^{(p+8)}(z)\ .
\end{aligned}
\tag{3.14}
$$

The expressions $A_k^p(z)$ are identically zero for odd integers k. Now, letting $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and evaluating each expression in (3.13) at $z = 0$, we obtain non-linear operators of degree 2, and, in particular, an expression for the elements of the generating sequence in terms of the real and imaginary parts of the zeros of the generating polynomial. Comparing coefficients, for the non-linear operator $a_k \mapsto \Lambda_k^1(\mu)$, we have $\mu_0 = (a^2+b^2), \mu_2 = (a^2-b^2)$. We observe that the situation described in Example 58 is not possible, as μ_0 is always positive for degree 2 operators generated by means of (3.13).

As with our previous attempts, even with the help of a computer the general case seems intractable; however, it is easy to see that the zeros of the generating polynomial must both be non-real. The following result may be viewed as an improvement on Corollary 55.

Theorem 59. *Let $g(z)$ be a quadratic polynomial with zeros $a \pm bi$. The non-linear operators $\Lambda^1(g) : a_k \mapsto \Lambda_k^1(g)$ and $\Lambda^2(g) : a_k \mapsto \Lambda_k^2(g)$ preserve $\mathscr{L}$-$\mathscr{P}^+$ with no restriction on the location of the non-real zeros of the generating polynomial. If the generating polynomial has only real zeros, the theorem holds provided the real zeros are all equal.*

Proof. Consider first the non-linear operator $a_k \mapsto \mu_0 a_k^2 + \mu_2 a_{k-1}a_{k+1}$. The coefficients $\mu_0 = (a^2+b^2), \mu_2 = (a^2-b^2)$ are obtained by direct calculation from (3.13). By Corollary 55, we need $h_1(x) = \mu_0 + x(\mu_0+\mu_2)$ to have all real and negative zeros. Substituting for the coefficients we obtain the polynomial $a^2+b^2+2b^2x$, which has a real negative zero provided $b \neq 0$; however, $b = 0$ produces the coefficients $\mu_0 = a^2, \mu_2 = a^2$, corresponding to the non-linear operator $a_k \mapsto a^2(a_k^2 + a_{k-1}a_{k+1})$, which preserves $\mathscr{L}$-$\mathscr{P}^+$ by Corollary 55.

Repeating the calculation for $a_k \mapsto \mu_0 a_k^2+\mu_2 a_{k-1}a_{k+1}+\mu_4 a_{k-2}a_{k+2}$ with the coefficients obtained from $A_4^1(z)$ (cf. (3.13)) we obtain the polynomials

$$\begin{aligned} h_2(x) &= 6a^2 + 12a^2b^2 + 6b^4 + (16a^4 + 24a^2b^2)x + 8a^4x^2, \text{ and} \\ \tilde{h}_2(x) &= 6a^2 + 12a^2b^2 + 6b^4 + (24a^4 + 24a^2b^2)x + 8a^4x^2, \end{aligned} \tag{3.15}$$

both of which have all real and negative zeros when $a \neq 0$. For the case when $a = 0$, we have $h_2(x) = \tilde{h}_2(x) = 6b^4$, corresponding to the non-linear operator $a_k \mapsto b^4\Lambda_k^2(g)$, which preserves $\mathscr{L}$-$\mathscr{P}^+$ by Theorem 42. □

Remark 60. Theorem 59 does not contradict Theorem 8 which requires the generating polynomial to be even with at least one non-real zero. The non-linear operator generated by the Cardon expressions (1.7) preserves $\mathscr{L}$-$\mathscr{P}^+$ even if the generating polynomial is not even and has all real non-distinct zeros (cf. Example 57). We make no claims about necessity or sufficiency (cf. Example 58).

Appealing to Theorem 59, if $\Lambda_k^1(g_1), \Lambda_k^1(g_2)$ are two non-linear stability preserving operators of order 2, then there is no restriction on the location of the zeros of the generating polynomials g_1 and g_2. If both g_1 and g_2 are even with non-real zeros, then adding the zeros pair-wise produces a conjugate pair of non-real numbers which we regard as the zeros of the polynomial which generates the operator $a_k \mapsto \Lambda_k^1(g_1) + \Lambda_k^1(g_2)$. If the zeros of both generating polynomials are all real, then they still lie in the same closed half-plane.

Definition 61. Let g_1, g_2 be real quadratic polynomials. We define $\oplus$ to be the pairwise addition of zeros of g_1 and g_2 that lie in the same half-plane so that the zeros of the real polynomial $g_1 \oplus g_2$ are a conjugate pair and we define $\odot$ to be the pairwise multiplication of zeros of g_1 and g_2 that lie in the same half-plane so that the zeros of the real polynomial $g_1 \odot g_2$ are a conjugate pair. □

Note that both $\oplus$ and $\odot$ are well-defined if the zeros of g_1 and g_2 are all non-real, or if the zeros of one polynomial are both real and the polynomial is even. If the zeros of one polynomial that is not even are both real and of opposite sign, then $\oplus$ is well defined, but $\odot$ may produce two different polynomials. Nevertheless, the resulting product will be a polynomial with a pair of conjugate zeros no matter how we multiply. When the zeros of one polynomial are both real and of the same sign, then both the operations $\oplus$ and $\odot$ are not-well defined, but still produce a polynomial with a conjugate pair of zeros every time.

We calculate the sum and product of the non-linear operators we have considered in Example 57. Observe that $x^2-\sqrt{2}x-1=(x-1-\sqrt{2})(x-1+\sqrt{2})$, so the operation $\odot$ produces two possible products.

Example 62. In terms of the generating polynomials, the sum of the operators in Example 57 is given by

$$\begin{aligned}\Lambda_k^1(2-\sqrt{2}x+x^2)+\Lambda_k^1(-1-2x+x^2) &= \Lambda_k^1(2-\sqrt{2}x+x^2\oplus -1-2x+x^2)\\ &= \Lambda_k^1\left(\frac{23}{4}+2\sqrt{3}-3x+x^2\right)\\ &= \frac{1}{4}(23+8\sqrt{3})a_k^2-\frac{1}{4}(5+8\sqrt{3})a_{k-1}a_{k+1}\ ,\end{aligned} \tag{3.16}$$

and its coefficients satisfy condition (i) of Theorem 55. Observe that the zeros of the generating polynomial $-1-2x+x^2$ are both real (cf. Remark 60), but of opposite signs, so $\oplus$ is well-defined. In terms of the generating polynomials, the product of the operators in Example 57 is given by

$$\begin{aligned}\Lambda_k^1(2-\sqrt{2}x+x^2)\cdot\Lambda_k^1(-1-2x+x^2) &= \Lambda_k^1(2-\sqrt{2}x+x^2\odot -1-2x+x^2)\\ &= \Lambda_k^1\left(6+4\sqrt{2}-(2+\sqrt{2})x+x^2\right)\\ &= (6+4\sqrt{2})a_k^2-(3+2\sqrt{2})a_{k-1}a_{k+1}\ ,\end{aligned} \tag{3.17}$$

if we use $1+\sqrt{2}$, and we obtain from $1-\sqrt{2}$ the operator

$$\begin{aligned}\Lambda_k^1(2-\sqrt{2}x+x^2)\cdot\Lambda_k^1(-1-2x+x^2) &= \Lambda_k^1(2-\sqrt{2}x+x^2\odot -1-2x+x^2)\\ &= \Lambda_k^1\left(6-4\sqrt{2}-(-2+\sqrt{2})x+x^2\right)\\ &= (6-4\sqrt{2})a_k^2-(3-2\sqrt{2})a_{k-1}a_{k+1}\ .\end{aligned} \tag{3.18}$$

By Theorem 55, both of the above products preserve $\mathscr{L}$-$\mathscr{P}^+$. $\square$

Restricting our attention to even generating polynomials with at least a pair of non-real zeros, as stipulated in Theorem 8, we have have the following corollary.

Corollary 63. *The class of non-linear operators of degree 2 and order 2 or 3, generated by even real polynomials, is a commutative monoid under the operations $\oplus$ and $\odot$ defined in Definition 61.* $\square$

3.2 Higher order operators

Let $f(z) \in \mathscr{L}\text{-}\mathscr{P}^+$ and let $g(z) = \prod_{k=1}^{m}(z-\alpha_k)$ be an even real polynomial with zeros $\alpha_1, \ldots, \alpha_m$. We wish to obtain an explicit forms of the Cardon expressions (1.7) $A_k(z)$ generated by means of the expression $\sum_{k=0}^{\infty} A_k(z)t^k = \prod_{j=1}^{m} f(z+\alpha_j t)$ (cf. Theorem 8). We will a need a standard lemma describing the k^{th} derivative of a product of m functions.

Lemma 64 ([32], [54])**.** *If $f_1, f_2, \ldots, f_m$ are functions with a sufficient number of derivatives, then*

$$\frac{d^k}{dt^k}\left\{\prod_{j=1}^{m} f_j(t)\right\} = \sum_{\lambda \vdash k} \frac{k!}{\lambda_1!\lambda_2!\cdots\lambda_m!} \prod_{j=1}^{m} \frac{d^{\lambda^j}}{dt^{\lambda_j}} f_j(t), \tag{3.19}$$

where $\lambda \vdash k$ means that the sum is taken over all integer partitions of k of length m; i.e., all m-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of non-negative integers such that $\sum_{j=1}^{m} \lambda_j = k$. $\square$

We follow D. Cardon and give an alternative representation of (1.7). An application of Lemma 64 to the Taylor coefficients

$$A_k(z) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\prod_{j=1}^{m} f(x+\alpha_j t)\right]\Bigg|_{t=0}, \tag{3.20}$$

yields the explicit formula,

$$A_k(z) = \sum_{\lambda \vdash k} \frac{1}{\lambda_1!\lambda_2!\cdots\lambda_m!} \prod_{j=1}^{m} \frac{d^{\lambda_j}}{dt^{\lambda_j}} f(z+\alpha_j t)\Bigg|_{t=0}, \tag{3.21}$$

where the sum is taken over all integer partitions of k of length m; i.e., all m-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of non-negative integers such that $\sum_{j=1}^{m} \lambda_j = k$. Alternatively, supposing that $f(z)$ is a polynomial of degree n with zeros $r_1, \ldots, r_n$, we have

$$\begin{aligned}\prod_{j=1}^{m} f(z+\alpha_j t) &= \prod_{j=1}^{m}\prod_{k=1}^{n}(z+\alpha_j t - r_k) \\ &= \prod_{k=1}^{n}\prod_{j=1}^{m}\left((z-r_k)+\alpha_j t\right).\end{aligned} \tag{3.22}$$

Next, using elementary symmetric functions of the zeros of $g(z)$,

$$\sum_{k=0}^{n} e_k(\alpha_1, \ldots, \alpha_n) z^{n-k} = \prod_{k=1}^{n} (z + \alpha_k), \tag{3.23}$$

and from (3.22), (3.23), and the assumption that $g(z+r_k) = \prod_{\ell=1}^{m} ((z + r_k) + \alpha_\ell) = \sum_{j=0}^{m} c_j (z+r_k)^j$ is an even polynomial,

$$\begin{aligned}\prod_{k=1}^{n} \prod_{j=1}^{m} ((z - r_k) + \alpha_j t) &= \prod_{k=1}^{n} \sum_{k=0}^{m} e_k(\alpha_1 t, \ldots, \alpha_m t)(z - r_k)^{m-k} \\ &= \prod_{k=1}^{n} \sum_{k=0}^{\frac{m}{2}} e_{2k}(\alpha_1, \ldots, \alpha_m)(z - r_k)^{m-2k} t^{2k} \\ &= \sum_{k=0}^{n \cdot \frac{m}{2}} A_{2k}(z) t^{2k}.\end{aligned} \tag{3.24}$$

Setting $f(z) = \sum_{k=0}^{\infty} a_k z^k$, evaluating $A_k(0)$, and shifting indices we arrive at the form of non-linear operators of various degrees and orders. For example, if the generating polynomial $g(z)$ has degree 2, then $A_2(z)$ corresponds to the partitions of 2 of length 2. Since $2 = 2 + 0 = 1 + 1$, we see that $A_2(0) = c_0 f''(z) f(z) + c_1 f'(z) f'(z)$ (cf. Theorem 8). It is not known, in general, if the non-linear operators obtained in this manner enjoy stability preserving properties.

Problem 65 (M. Chasse)**.** *If a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ has only real non-positive zeros, and if for $k = 0, 1, 2, \ldots, n$,*

$$b_k := -a_{k-2}^3 a_{k+2} + 3a_{k-2}^2 a_k^2 + 6a_{k-1}^4 + 4a_{k-2}^2 a_{k+1} a_{k-1} - 12 a_{k-2} a_{k-1}^2 a_k, \tag{3.25}$$

where $a_k = 0$ for $k < 0$ and $k > n$, then must $q(x) = \sum_{k=0}^{n} b_k x^k$ have only real non-positive zeros? □

A positive answer to Problem 65 could indicate an alternative method of proof of Theorem 37. We observe that the expression (3.25) can be obtained from Theorem 8 by setting the generating polynomial to be $z^4 + 1$. This produces the Cardon expression of degree 4 and order 5,

$$12f'(x)^4 - 24f(x)f'(x)^2 f''(x) + 6f(x)^2 f''(x)^2 + 8f(x)^2 f'(x) f'''(x) - 2f(x)^3 f^{(4)}(x), \tag{3.26}$$

which, evaluated at $x = 0$, yields the coefficient-wise transformation $2b_k$ above.

3.2.1 Determinant representations

By Theorem 42 the sequences $\{\Lambda_k^p(z^2+1)\}_{k=0}^{\infty}$ are totally positive sequences for each positive integer p, provided that the non-linear operators defined by $a_k \mapsto \Lambda_k^p(z^2+1)$ are acting on functions in $\mathscr{L}\text{-}\mathscr{P}^+$.

For positive integers r, the non-linear operator S_r, in the determinant form

$$a_k \mapsto \begin{vmatrix} a_k & a_{k-r} \\ a_{k+r} & a_k \end{vmatrix}, \tag{3.27}$$

occurs naturally as a minor of the infinite Toeplitz matrix $A = (a_{ij})$, obtained from the coefficients of $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+$, by setting $a_{ij} = a_{k+(i-j)}$:

$$A = \begin{pmatrix} & \vdots & \vdots & \vdots & & \vdots & \\ \cdots & a_k & a_{k-1} & a_{k-2} & \cdots & a_{k-r} & \cdots \\ \cdots & a_{k+1} & a_k & a_{k-1} & \cdots & a_{k-r+1} & \cdots \\ \cdots & a_{k+2} & a_{k+1} & a_k & \cdots & a_{k-r+2} & \cdots \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \\ \cdots & a_{k+r} & a_{k+r-1} & a_{k+r-2} & \cdots & a_k & \cdots \\ & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}. \tag{3.28}$$

With the exception of $\Lambda^1(z^2+1) = S_1$, it seems that the non-linear operators $a_k \mapsto \Lambda_k^p(z^2+1)$ cannot be realized as a minor of the above matrix A. From (2.38) and the Laplace expansion along the i^{th} row of the determinant of a $p \times p$ submatrix $B := (b_{ij})$ of A,

$$\det B = \sum_{j=1}^{p} (-1)^{j+i} b_{ij} \det B(i|j), \tag{3.29}$$

where $B(i|j)$ is the submatrix of B obtained by deleting row i and column j, it is clear that $\Lambda_k^p(z^2+1)$ is not the determinant of any submatrix B of A larger than 2×2. Moreover, for each positive integer p, the non-linear operator Λ^p consists of the term $\binom{2p-1}{p} a_k^2$ and p other terms, and for $p > 1$ cannot be realized as a 2×2 determinant, and *a fortiori*, any minor of A larger than 2×2.

Let $\tilde{A} = (\tilde{a}_{ij})$ be the infinite Toepliz matrix obtained from the coefficients of $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k \in$

$\mathscr{L}$-$\mathscr{P}^+$, by setting $\tilde{a}_{ij} = \gamma_{k+(i-j)}$ and let $\tilde{B}$ be the principal submatrix

$$\begin{pmatrix} \gamma_k & \gamma_{k-1} & \gamma_{k-2} & \gamma_{k-3} \\ \gamma_{k+1} & \gamma_k & \gamma_{k-1} & \gamma_{k-2} \\ \gamma_{k+2} & \gamma_{k+1} & \gamma_k & \gamma_{k+1} \\ \gamma_{k+3} & \gamma_{k+2} & \gamma_{k+1} & \gamma_k \end{pmatrix}. \tag{3.30}$$

We list the known higher order Laguerre-Turán coefficient inequalities for function in the Laguerre-Pólya class.

Table 3.1: Higher Order Coefficient Inequalities of $\varphi(x) = \sum_{k=0}^{\omega} \frac{\gamma_k}{k!} x^k (0 \le \omega \le \infty)$

#	Conditions	Inequality	Notes
1	$\varphi(x) \in \mathscr{L}$-$\mathscr{P}$	$4(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) - (\gamma_k\gamma_{k+1} - \gamma_{k-1}\gamma_{k+2})^2 \ge 0$ $4\det(\tilde{B}(3,4\|3,4)\tilde{B}(1,4\|3,4)) - (\det(\tilde{B}(2,4\|3,4)))^2$	[33] determinant form
2	$\varphi(x) \in \mathscr{L}$-$\mathscr{P}^+$	$2\gamma_{k+1}(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) - \gamma_k(\gamma_{k+1}\gamma_{k+2} - \gamma_k\gamma_{k+2}) \ge 0$ $2\gamma_{k+1}\det(\tilde{B}(1,4\|3,4)) - \gamma_k \det(\tilde{B}(1,3\|3,4))$	[21] determinant form

3.2.2 Toeplitz-type non-linear operators

We are motivated by a question S. Fisk ([36, Question 3]) and Brändén's Theorem (Theorem 35) to consider the non-linear operators that are represented by the principal minors of the Toepliz matrix $A = (a_{ij})$ obtained from $\sum_{k=0}^{\infty} a_k x^k$ by setting $a_{ij} = a_{k+(i-j)}$. The non-linear operator $\Lambda^1(z^2+1) : a_k \mapsto a_k^2 - a_{k-1}a_{k+1}$ is exactly $T^2 : a_k \mapsto \begin{vmatrix} a_k & a_{k-1} \\ a_{k+1} & a_k \end{vmatrix}$. We will call these operators $T^p, p \ge 2$, where p represents the order of the principal minor of A we have used.

The natural question at this point is to consider the non-linear operator T^3, given by

$$a_k \mapsto \begin{vmatrix} a_k & a_{k-1} & a_{k-2} \\ a_{k+1} & a_k & a_{k-1} \\ a_{k+2} & a_{k+1} & a_k \end{vmatrix}, \tag{3.31}$$

and its stability preserving properties. To this end, we list some relationships between Toeplitz-type operators that will be useful in the sequel. In Section 4.1.6 we will return to the stability preserving properties of T^3 and consider Hadamard products involving the T^3 operator. The non-linear operator T^3 is not in general a stability preserver; however, R. Yoshida (see [91]) showed that, for positive integers p and n, the non-linear operator T^p preserves $\mathscr{L}$-$\mathscr{P}^+$ when it acts on e^x and $(1+x)^n$.

Proposition 66. *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of positive numbers. Then,*

(i) $T^2[T^2[a_k]] = a_k \cdot T^3[a_k]$;

(ii) $T^2[T^2[T^2[a_k]]] = T^2[a_k] \cdot T^3[T^2[a_k]]$;

(iii) $T^2[T^2[T^2[T^2[a_k]]]] = a_k^4 \cdot T^3[a_k] \cdot T^3[T^3[a_k]]$.

Proof. A direct, albeit very tedious calculation, verifies each of the above properties. In general, the T^p operator acts on the coefficients a_k. Hence, to check property (iii), we operate with T^2 on both sides of (ii),

$$\begin{aligned} T^2[T^2[T^2[T^2[a_k]]]] &= T^2[T^2[a_k] \cdot T^3[T^2[a_k]]] \\ &= T^2[T^2[a_k]] \cdot T^3[T^2[T^2[a_k]]] \\ &= a_k \cdot T^3[a_k] \cdot T^3[a_k \cdot T^3[a_k]] \\ &= a_k \cdot T^3[a_k] \cdot a_k^3 \cdot T^3[T^3[a_k]] \\ &= a_k^4 \cdot T^3[a_k] \cdot T^3[T^3[a_k]] \,. \end{aligned} \tag{3.32}$$

□

Remark 67. Note that item (i) of Proposition 66 appears in the proof of Corollary 2.14 in [18].

According to our classification of non-linear operators by means of generating polynomials, the $T^p, p \geq 2$, operators are degree p. However, $T^3 : a_k \mapsto a_k^3 - 2a_{k-1}a_k a_{k+1} + a_{k-1}a_{k+1}^2 + a_{k-1}^2 a_{k+2} - a_{k-2}a_k a_{k+2}$ is not generated by a polynomial of degree 3, because it is of the wrong order. Only the integer 5 can be partitioned into partitions of 3 non-negative integers in 5 different ways.

Lemma 68. *The non-linear operator T^3 preserves multiplier sequences of the first kind.*

Proof. Let $\varphi(x) = \sum_{k=0}^{\omega} a_k x^k \in \mathscr{L}$-$\mathscr{P}^+, 0 \leq \omega \leq \infty$. Then, by Theorem 42, $T^2[T^2[\varphi(x)]] \in \mathscr{L}$-$\mathscr{P}^+$, but by Proposition 66,

$$\sum_{k=0}^{\omega} T^2[T^2[a_k]]x^k = \sum_{k=0}^{\omega} T^3[a_k] \cdot a_k x^k \,. \tag{3.33}$$

□

A natural question at this point, to advance progress toward the solution of S. Fisk's problem ([36, Question 3]), is to ask when, if at all, the a_k may be removed from (3.33). We return to this question in §4.1.6, where we consider the Hadamard products of Toeplitz-type operators.

CHAPTER 4
APPLICATIONS AND EXAMPLES

4.1 Highly log-concave sequences

4.1.1 Introduction

The sequence of coefficients $\{a_k\}_{k=0}^n$ of a real polynomial $P_n(x) = \sum_{k=0}^n a_k z^k$, all of whose zeros are real, satisfies various concavity conditions, the principal of which are due to I. Newton (see [47, §4.3]); i.e., for $k = 1, 2, \ldots, n-1$,

$$a_k^2 - \frac{n-k+1}{n-k}\frac{k+1}{k}a_{k-1}a_{k+1} \geq 0. \tag{4.1}$$

Definition 69. A sequence $\{a_k\}_{k=0}^{\omega}, 0 \leq \omega \leq \infty$, of real positive numbers is said to be *log-concave* if for $k = 1, 2, 3, \ldots, \omega - 1$, or for all positive integers k in the case of an infinite sequence, the inequalities

$$a_k^2 - a_{k-1}a_{k+1} \geq 0, \tag{4.2}$$

all hold. □

Commonly encountered variants of log-concavity are *strong log-concavity*, the log-concavity of the sequence $\{k!a_k\}_{k=0}^{\infty}$, and *ultra log-concavity of order m*, the log-concavity of the sequence $\left\{\frac{a_k}{\binom{m}{k}}\right\}_{k=0}^{m}$. A routine calculation shows that (4.1) holds for $\{a_k\}_{k=1}^{n-1}$ if and only if the sequence $\{a_k\}_{k=0}^{n}$ is an ultra log-concave sequence. A routine verification shows that ultra log-concavity implies strong log-concavity, which in turn implies log-concavity. It is also easy to check that log-concavity of the sequence of positive numbers $\{a_k\}_{k=0}^{\infty}$ is equivalent to the statement that the sequence $\{\frac{a_k}{a_{k+1}}\}_{k=0}^{\infty}$ is non-decreasing. A stronger condition is that $\{a_k\}_{k=0}^{\infty}$ is TP_2 (see Definition 27). If the sequence of positive numbers $\{a_k\}_{k=0}^{\infty}$ is TP_2 (cf. Theorem 12), then the associated sequence $\{\frac{a_k}{a_{k+1}}\}_{k=0}^{\infty}$ is non-decreasing. The classical results on concavity of coefficients are often stated as the following folk theorem. Note that the emphasis is now on real entire functions all of whose coefficients are positive.

Theorem 70 ([10, Theorem 1.2.1])**.** *Let $\sum_{k=0}^{d} a_k x^k$ be a polynomial with non-negative coefficients and with only real zeros. Then the sequence $\{a_0, a_1, \ldots, a_d\}$ is log-concave.* □

A partial converse to Theorem 70 is given in [89], and among the many generalizations of Theorem 70 we have another classical result relating the location of the zeros of an entire function to the log-concavity of its coefficient sequence.

Theorem 71 ([12])**.** *Let $G(z)$ be a real entire function with the product and series representations*

$$G(z) = \prod_k (1+\rho_k z) = \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}, \tag{4.3}$$

and suppose that $0 \leq |\Im \rho_k| < \Re(\rho_k)$ for all k. Then the Turán inequalities $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$ hold for for all $k \geq 1$ if $G(z)$ has infinitely many roots and for $1 \leq k \leq d$ if $G(z)$ is a polynomial of degree d. □

Owing to the fact that the primary motive for our study of highly log-concave sequences is to better understand real entire functions whose coefficients form highly log-concave sequences, we will use the terminology of this section also when describing real entire functions. Following S. Karlin (cf. [55, p. 393]), we will call $\sum_{k=0}^{\infty} a_k x^k$ the *generating function* of the sequence of coefficients $\{a_k\}_{k=0}^{\infty}$. We will often work with real entire functions whose coefficients sequences are log-concave sequences, though this property is not, in general, inherited from the log-concavity of the generating function, which we define next.

Definition 72. A continuous function $f : (a,b) \to (0,\infty)$, $-\infty \leq a < b \leq \infty$, is called *log-concave* on (a,b) if for any $\delta > 0$ and μ such that $[\mu-\delta, \mu+\delta] \subset (a,b)$,

$$(f(\mu))^2 \geq f(\mu-\delta)f(\mu+\delta). \tag{4.4}$$

If inequality (4.4) is reversed, then f is called *log-convex* on (a,b). □

Log-concavity of a function g is simply the concavity of the logarithm of g. Indeed, a function $g(x)$ which satisfies the inequality

$$g\left(\frac{x+y}{2}\right) \geq \frac{g(x)+g(y)}{2}\ , \tag{4.5}$$

for all $x > y$ in an interval is said to be concave on that interval (see [47, §3.5]). If the logarithm of a function f is concave on an interval, setting $\mu = \frac{x+y}{2}$, $\delta = \frac{x-y}{2}$, and $g(x) = \ln f(x)$ in (4.5) produces (4.4).

In order to avoid confusion, we will not call a function whose sequence of coefficients is log-concave a log-concave function. We hasten to add that this practice is neither accidental nor abusive when specifically referring to the Laguerre-Pólya class. Indeed, real entire function $f(x) = \sum_{k=0}^{\omega} a_k x^k, 0 \leq \omega \leq \infty$, is log-concave on $\mathbb{R}$ if and only if $\frac{d^2}{dx^2} \ln f(x) \leq 0$ for all $x \in \mathbb{R}$ (see [47, §3.10]). It follows that for all $x \in \mathbb{R}$, we have the inequality $(f'(x))^2 - f'(x)f''(x) \geq 0$. In particular, when $x = 0$, we obtain the familiar Turán inequality for $k = 1$, $a_1^2 - a_0 a_2 \geq 0$. Specializing to the Laguerre-Pólya class, which is closed under differentiation, we may repeat the calculation for $f'(x)$ to obtain $a_2^2 - a_1 a_3 \geq 0$, and proceed thus for higher derivatives of $f(x)$ to conclude that the sequence of coefficients of a real entire function $f(x) = \sum_{k=0}^{\omega} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+, 0 \leq \omega \leq \infty$, is a log-concave sequence. In other words, in the Laguerre-Pólya class the coefficient sequence inherits log-concavity from its generating function. Conversely, the complete characterization of the Laguerre-Pólya class given in Theorem 11 depends on the condition that the sequence $\{k! a_k\}_{k=0}^{\omega}$ be log-concave, which implies that the sequence $\{a_k\}_{k=0}^{\omega}$ is log-concave. However, a log-concave function need not be the generating function of a log-concave sequence, nor *vice versa*, as illustrated in the following example.

Example 73. The polynomial $x^2 + 2x + 2$ generates the log-concave sequence $\{2, 2, 1\}$; however, a calculation shows that

$$\frac{d^2}{dx^2} \ln\left((x+1)^2 + 1\right) = -\frac{2x(x+2)}{(2 + 2x + x^2)^2}, \tag{4.6}$$

is in fact positive on the interval $(-2, 0)$. On the other hand, the sequence $\{1, 3, 2, 1, 1\}$ is not log-concave, but it can be checked with the aid of a computer that

$$\frac{d^2}{dx^2} \ln\left(1 + 3x + 2x^2 + x^3 + x^4\right) = -\frac{5 + 6x - 4x^2 - 4x^3 + 7x^4 + 6x^5 + 4x^6}{(1 + 3x + 2x^2 + x^3 + x^4)^2}, \tag{4.7}$$

is in fact negative on its domain. $\square$

It will be useful to introduce at this point terminology relating log-concavity and the Turán inequalities. The following nomenclature is often found in the vast literature on the subject of log-concavity and, in accord with well-established traditions, we adopt it in our work, but not before making some adjustments.

Definition 74. A sequence of positive numbers $\{a_k\}_{k=0}^{\omega}, 0 \leq \omega \leq \infty$, that for $k = 1, 2, 3, \dots, \omega - 1$, or for all positive integers k in case of an infinite sequence, satisfies the Turán-type inequalities

$$a_k^2 - \alpha a_{k-1} a_{k+1} \geq 0, \tag{4.8}$$

will be called a *α-log-concave* sequence. When $\alpha = 1$ we will, for simplicity, call a 1-log-concave sequence log-concave, and will call a α-log-concave sequence *highly log-concave*, whenever $\alpha > 1$. The constant $\alpha := \inf \left\{ \frac{a_k^2}{a_{k-1}a_{k+1}} \right\}_{k=1}^{\omega}$ will be called the *Turán constant* associated to the sequence $\{a_k\}_{k=0}^{\omega}$. □

In the sequel we will study the conditions under which non-linear operators encountered in Chapter 3 preserve the log-concavity of coefficients sequences of real entire functions. We recall that the convolution of log-concave sequences is log-concave (see [31], [55, p. 394]) and the convolution of ultra log-concave sequences is log-concave (see [66]). A routine calculation also confirms our intuition that the Hadamard product of log-concave sequences is log-concave, i.e., under the Hadamard product the class of log-concave sequences forms a semi-group.

4.1.2 Rapidly decreasing sequences

The work of E. Laguerre on uniform limits of polynomials with only real zeros motivated M. Petrovitch, among others, to study entire functions $\sum_{k=0}^{\infty} a_k x^k$ all of whose partial sums $\sum_{k=m}^{n} a_k x^k, 0 \leq m \leq n \leq \infty$, have only real zeros. The following classical result of J. I. Hutchinson, rediscovered in [63], characterizes a class of entire functions belonging to the Laguerre-Pólya class.

Theorem 75 ([51])**.** *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a real entire function, all of whose coefficients are positive. The condition that the coefficients of $f(x)$ satisfy the inequalities $a_k^2 - 4a_{k-1}a_{k+1} \geq 0$, for all positive integers k, is necessary and sufficient for the zeros of $f(x)$ and all of its partial sums $\sum_{k=m}^{n}, 0 \leq m < n < \infty$, to all be real, simple, and negative, except possibly at $x = 0$.* □

In light of Hutchinson's Theorem, we distinguish from the class of highly log-concave sequences the class of α-log-concave sequences in the case when $\alpha \geq 4$.

Definition 76. A sequence $\{a_k\}_{k=0}^{\omega}, 0 \leq \omega \leq \infty$, of positive real numbers that for $k = 1, 2, 3, \dots, \omega-1$, or for all positive integers k in case of an infinite sequence, satisfies the Turán inequalities with Turán constant at least 4 is called a *rapidly decreasing sequence.* □

According to our Definitions 74 and 76, a highly log-concave sequence is log-concave and a rapidly decreasing sequence is a sequence of positive numbers that is at least 4-log-concave. We remark that in their 1995 study of complex zero decreasing sequences [19], T. Craven and G. Csordas defined a rapidly decreasing sequence to be any sequence $\{a_k\}_{k=0}^{\infty}$ of positive real numbers satisfying the Turán inequalities with Turán constant α^2, where

$$\alpha \geq \max\left\{2, \frac{\sqrt{2}}{2}\left(1+\sqrt{1+a_1}\right)\right\} . \tag{4.9}$$

We depart from this approach of associating the log-concavity of a sequence with any one particular term. In the sequel we will study sequences of positive numbers that satisfy the Turán inequalities with Turán constant $1 < \alpha < 4$, and restrict our attention primarily to non-increasing sequences. The analogous situation, when the coefficient sequence is non-decreasing, has been studied extensively. We recall a classical result on the location of zeros by means of coefficient inequalities.

Theorem 77 ([67, Eneström-Kakeya])**.** *If $0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$, then all of the zeros of the polynomial $\sum_{k=0}^{n} a_k z^k$ lie in the disk $|z| \leq 1$.*

The Eneström-Kakeya Theorem has been improved and extended in various directions. In particular, analogous results are known for other bases, for instance the Hermite basis. We refer to [79] for a detailed study and recent results in this area. Recent developments in the study of the Turán constant α and its relationship to Hutchinson's Theorem may be pursued in [46], [59], [61], and the references contained therein. Of particular interest in stability is the next result, relating the Turán constant to Hurwitz stability.

Theorem 78 ([59])**.** *In order for a real entire function to be Hurwitz stable it is sufficient that its sequence of coefficients is positive and $\sqrt{x_0}$-log-concave, where $x_0 \approx 2.1479$ is the positive zero of $x^3 - x^2 - 2x - 1$.* □

We end this section with a classical result on the order of entire functions whose coefficients form a rapidly decreasing sequence.

Proposition 79 ([51])**.** *Let $\{a_k\}_{k=0}^{\infty}$ be a real sequence of positive numbers that is at least α^2-log-concave. Then,*

$$a_k \leq \frac{a_0}{\alpha^{k(k-1)}}\left(\frac{a_1}{a_0}\right)^k, \; k = 1, 2, 3, \ldots . \tag{4.10}$$

Proof. Let $\{a_k\}_{k=0}^{\infty}$ be a real sequence of positive numbers and suppose that $a_{k+1}^2 \geq \alpha^2 a_k a_{k+2}, k = 1, 2, 3, \ldots$. Multiplying the first k of these Turán-type inequalities , we obtain

$$\begin{aligned} a_1^2 a_2^2 a_3^2 \cdots a_{k-1}^2 a_k^2 &\geq (\alpha^2)^k a_0 a_1 a_1 a_3 a_2 a_4 \cdots a_{k-2} a_k a_{k-1} a_{k+1} \\ a_1 a_k &\geq (\alpha^2)^k a_0 a_{k+1}. \end{aligned} \tag{4.11}$$

Next, we multiply together the first k of the inequalities $\frac{a_{k+1}}{a_k} \leq \frac{a_1}{a_0} \frac{1}{(\alpha^2)^k}$, and obtain

$$\begin{aligned} \frac{a_1}{a_0} \frac{a_2}{a_1} \frac{a_3}{a_2} \cdots \frac{a_{k-1}}{a_{k-2}} \frac{a_k}{a_{k-1}} &\leq \left(\frac{a_1}{a_0}\right)^k \left(\frac{1}{\alpha^2}\right)^{\sum_{j=0}^{k-1} j} \\ \frac{a_k}{a_0} &\leq \left(\frac{a_1}{a_0}\right)^k \left(\frac{1}{\alpha^2}\right)^{\frac{k(k-1)}{2}}, \end{aligned} \tag{4.12}$$

whence $a_k \leq a_0 \left(\frac{a_1}{a_0}\right)^k \frac{1}{\alpha^{k(k-1)}}$. □

By means of a number of well-known formulas relating the order of an entire function (see for example [3], [68]) to its Taylor coefficients, and the above estimate, we see that an entire function whose coefficients form a rapidly decreasing sequence is necessarily of order zero. Moreover, by Hutchinson's Theorem, a real entire function whose coefficients form a rapidly decreasing sequence belongs to $\mathscr{L}$-$\mathscr{P}^+$. As a corollary, we state a useful observation regarding the generating function of a non-increasing highly log-concave sequence.

Corollary 80. *Let $\{a_k\}_{k=0}^{\infty}$ be a non-increasing highly log-concave sequence of real numbers. Suppose that in addition to the condition that $a_{k-1} \geq a_k$, for all positive integers k, also the inequalities $\beta a_k \geq \alpha a_{k-1}$ hold for some $\beta > \alpha > 1$. Then the generating function of $\{a_k\}_{k=0}^{\infty}$, $\sum_{k=0}^{\infty} a_k x^k$, is entire.*

Proof. The sequence $\{a_k\}_{k=0}^{\infty}$ satisfies, for some $\alpha > 1$ and all positive integers k the inequalities $a_k^2 \geq \alpha a_{k-1} a_{k+1}$, and we also have $a_0 \leq \frac{\beta}{\alpha} a_1$ and $a_0 \geq a_1$. An application of Proposition 79 yields the estimate,

$$\begin{aligned} a_k &\leq a_0 \left(\frac{a_1}{a_0}\right)^k \frac{1}{\alpha^{\frac{k(k-1)}{2}}} \\ &\leq \frac{\beta}{\alpha} a_1 \left(\frac{a_1}{a_0}\right)^k \frac{1}{\alpha^{\frac{k(k-1)}{2}}} \\ &\leq \frac{\beta a_1}{\alpha^{\frac{k(k-1)}{2}+1}}, \end{aligned} \tag{4.13}$$

and whence $\overline{\lim}_{k\to\infty} |a_k|^{\frac{1}{k}} = \beta a_1 \overline{\lim}_{k\to\infty} \frac{1}{\alpha^{\frac{(k-1)}{2}+1}} = 0.$ □

We note that if $\alpha = \beta$ in Corollary 80, then the sequence $\{a_k\}_{k=0}^{\infty}$ is constant. Constant sequences of positive real numbers are log-concave as each Turán ratio is equal to 1, but not highly log-concave. More importantly, a constant sequence cannot be generated by a transcendental entire function.

4.1.3 Action of non-linear operators on highly log-concave sequences

We will study the conditions under which non-linear operators preserve the log-concavity of coefficients sequences of real entire functions. In conjunction with results relating the log-concavity of the coefficient sequences to the location of zeros of real entire functions, we obtain results on the reality and stability preserving properties of certain non-linear operators. We begin with the study of the class S_r of non-linear operators (cf. §2.2.1). This non-linear operator is of particular interest, because it is a natural building block of the non-linear operators $\Lambda^p(z^2+1) : a_k \mapsto \Lambda_k^p(z^2+1)$ (Proposition 49). R. Yoshida asked [91, Open Problem 5] whether the non-linear operator $S_r : a_k \mapsto a_{k-r}a_{k+r}$ preserves the class of rapidly decreasing sequences as defined by T. Craven and G. Csordas. The investigation of this question led us to the following elementary result, which appears, however, to be new.

Theorem 81. *Let $\{a_k\}_{k=0}^{\omega}, 0 \le \omega \le \infty$, be a real sequence with terms that alternate in sign or have the same sign. Suppose that for a fixed $\alpha > 0$ and all positive integers k, the inequalities $a_k^2 - \alpha^2 a_{k-1}a_{k+1} \ge 0$ and $|a_k| \ge |a_{k+1}|$ hold. Then, for any positive integer r and all positive integers k, the sequence $\{b_{k,r} := a_k^2 - a_{k-r}a_{k+r}\}_{k=0}^{\omega}$ satisfies the inequalities $b_{k,r}^2 - \alpha^4 b_{k-1,r}b_{k+1,r} \ge 0$, provided that $\alpha^4 \ge 2$.*

Proof. We may assume that $k - r \ge 0$, else according to our conventions $a_{k-r} := 0$ (cf. Notation 36), and there is nothing to prove. Observe that products of the form $a_{k-m}a_{k+m}, m = 0, 1, 2, \ldots,$ are positive. Let r be a positive integer and set $b_{k,r} := a_k^2 - a_{k-r}a_{k+r}$. Then,

$$\begin{aligned} b_{k,r}^2 &= a_k^4 - 2a_k^2 a_{k-r}a_{k+r} + a_{k-r}^2 a_{k+r}^2 \\ &\ge a_k^4 - 2a_k^2 a_{k-r}a_{k+r} + (\alpha^2 a_{k-r-1}a_{k-r+1})(\alpha^2 a_{k+r-1}a_{k+r+1})\,. \end{aligned} \tag{4.14}$$

Next,

$$\begin{aligned} b_{k-1,r}b_{k+1,r} &= (a_{k-1}^2 - a_{k-r-1}a_{k+r-1})(a_{k+1}^2 - a_{k-r+1}a_{k+r+1}) \\ &= a_{k-1}^2 a_{k+1}^2 - a_{k-1}^2 a_{k-r+1}a_{k+r+1} - a_{k+1}^2 a_{k-r-1}a_{k+r-1} \\ &\quad + a_{k-r-1}a_{k+r-1}a_{k-r+1}a_{k+r+1} \ , \end{aligned} \tag{4.15}$$

and substituting for $a_{k-r-1}a_{k+r-1}a_{k-r+1}a_{k+r+1}$ in (4.14), yields

$$\begin{aligned} b_{k,r}^2 &\geq \alpha^4 b_{k-1,r}b_{k+1,r} + a_k^4 - \alpha^4 a_{k-1}^2 a_{k+1}^2 \\ &\quad + \alpha^4 a_{k-1}^2 a_{k-r+1}a_{k+r+1} + \alpha^4 a_{k+1}^2 a_{k-r-1}a_{k+r-1} - 2a_k^2 a_{k-r}a_{k+r} \\ &\geq \alpha^4 b_{k-1,r}b_{k+1,r} + \alpha^4 a_{k+1}^2 a_{k-r-1}a_{k+r-1} - 2a_k^2 a_{k-r}a_{k+r} \ . \end{aligned} \tag{4.16}$$

Note that in (4.16) the expression $a_k^4 - \alpha^4 a_{k-1}^2 a_{k+1}^2$ is non-negative by hypothesis and $\alpha^4 a_{k-1}^2 a_{k-r+1}a_{k+r+1}$ is positive. We now need to guarantee $\alpha^4 a_{k+1}^2 a_{k-r-1}a_{k+r-1} - 2a_k^2 a_{k-r}a_{k+r} \geq 0$. By hypothesis, $|a_k| \geq |a_{k+1}|$ for all indices k, hence

$$\begin{aligned} \alpha^4 a_{k+1}^2 a_{k-r-1}a_{k+r-1} - 2a_k^2 a_{k-r}a_{k+r} &\geq \alpha^4 a_{k+1}^2 a_{k-r}a_{k+r} - 2a_k^2 a_{k-r}a_{k+r} \\ &= a_{k-r}a_{k+r}(\alpha^4 a_{k+1}^2 - 2a_k^2) \ . \end{aligned} \tag{4.17}$$

Forcing $\alpha^4 a_{k+1}^2 - 2a_k^2 \geq 0$ is equivalent to requiring $a_{k+1}^2 \geq \frac{2}{\alpha^4} a_k^2$, where $|a_k| \geq |a_{k+1}|$ also forces $\alpha^4 \geq 2$. □

Applying Theorem 75 yields this immediate corollary.

Corollary 82. *If the coefficients of $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+$ form a non-increasing sequence and satisfy the Turán inequalities with Turán constant 4, then for any positive integer r, $S_r[f(x)] = \sum_{k=0}^{\infty}(a_k^2 - a_{k-r}a_{k+r})x^k \in \mathscr{L}\text{-}\mathscr{P}^+$. In particular, the S_r operator preserves the class of rapidly decreasing sequences.* □

We remark that even though the non-linear operators S_r may be used to obtain the non-linear operators Λ^p (cf. Proposition 49), we may not conclude from Theorem 81 that the non-linear operators $\Lambda^p(z^2+1)$ enjoy similar log-concavity-preserving properties. In part, the reason for this is that in general log-concavity of the generating function of a sequence is not additive, even though log-convexity is additive.

If we restrict our attention to polynomials, the role of the parameter r in the non-linear operator

S_r in the proof of Theorem 81 is merely for convenience. By our standing conventions (cf. Notation 36), there is nothing to prove if the choice of r causes all terms of the form $a_{k-r}a_{k+r}$ to vanish. The log-concavity of the coefficient sequence $\{a_k\}_{k=0}^n$ of a real polynomial follows immediately from the log-concavity of the sequence $\{a_k^2\}_{k=0}^n$.

Theorem 81 guarantees that if the sequence $\{a_k\}_{k=0}^\infty$ is *at least* rapidly decreasing in the sense of Hutchinson's Theorem; i.e., the Turán inequalities with Turán constant at least 4 hold, then the sequence $\{a_k^2 - a_{k-r}a_{k+r}\}_{k=0}^\infty$ will be at least rapidly decreasing. Iterating thus, we obtain an infinite sequence of multiplier sequences. For a detailed study of iterated Turán inequalities we refer to [19], [20], [21], and the references contained therein. The sequence of iterated Turán inequalities obtained from a rapidly decreasing sequence of positive numbers is known to be rapidly decreasing. The following theorem makes this notion precise (see also [69]).

Theorem 83 ([21, Theorem 4.1])**.** *Fix $c \geq 1$ and $d \geq 0$. Consider the set $\mathscr{M}_c$ of all sequences of positive numbers $\{\gamma_k\}_{k=0}^\infty$ satisfying the Turán inequality with Turán constant c. Then the sequence $\{\gamma_k^2 - \gamma_{k-1}\gamma_{k+1}\}_{k=0}^\infty$ satisfies the Turán inequality with Turán constant $c+d$ for all k and all sequences in $\mathscr{M}_c$ if and only if $c \geq \frac{3+\sqrt{5+4d}}{2}$.* □

Specializing Theorem 81 to the non-linear operator S_1, we obtain the following corollary.

Corollary 84. *Let $\{a_k\}_{k=0}^\infty$ be a real sequence of positive numbers. If $a_k > a_{k+1}$ for $k = 0, 1, 2, \ldots$, and $\{a_k\}_{k=0}^\infty$ is α-log-concave for some $\alpha \geq 1 + \sqrt{2}$, then $\{a_k^2 - a_{k-1}a_{k+1}\}_{k=0}^\infty$ is also α-log-concave.*

Proof. Let $\{a_k\}_{k=0}^\infty$ be a real sequence of positive numbers. Suppose that $a_k > a_{k+1}$ for $k = 0, 1, 2, \ldots$, and that for some $\alpha \geq \sqrt{2}$ the sequence is α-log-concave. Applying Theorem 83 we see that the α-log-concavity of the sequence $\{a_k\}_{k=0}^\infty$ is equivalent, for some $d \geq 0$, to the $(\alpha + d)$-log-concavity of the sequence $\{a_k^2 - a_{k-1}a_{k+1}\}_{k=0}^\infty$ if and only if $\alpha \geq \frac{3+\sqrt{5+4d}}{2}$. Hence, we obtain an upper bound for d

$$\begin{aligned} \alpha \geq \frac{3+\sqrt{5+4d}}{2} &\iff 2\alpha - 3 \geq \sqrt{5+4d} \\ &\iff (2\alpha-3)^2 \geq 5 + 4d \\ &\iff 4\alpha^2 - 12\alpha + 4 \geq 4d \\ &\iff \alpha^2 - 3\alpha + 1 \geq d. \end{aligned} \tag{4.18}$$

By Theorem 81, the sequence $\{a_k^2 - a_{k-1}a_{k+1}\}_{k=0}^\infty$ is α^2-log-concave, whence a lower bound on the

Turán constant sufficient for the $(\alpha+d)$-log-concavity of the sequence $\{a_k^2 - a_{k-1}a_{k+1}\}_{k=0}^{\infty}$ is 2. With the lower bound $\alpha + d \geq 2$ and the above upper bound on d, we have

$$\begin{aligned} \alpha^2 - 3\alpha + 1 \geq d \geq 2 - \alpha &\Rightarrow \alpha^2 - 2\alpha - 1 \geq 0 \\ &\Rightarrow \alpha \geq 1 + \sqrt{2}. \end{aligned} \tag{4.19}$$

□

Observe that $1+\sqrt{2} < \frac{3+\sqrt{5}}{2}$ and so the assumption that $\{a_k\}_{k=0}^{\infty}$ is a strictly decreasing sequence is necessary, otherwise we would be contradicting Theorem 83. We recall that real polynomials whose coefficient sequences are non-decreasing sequences have their zeros confined to the unit disk (Theorem 77). Non-linear operators that preserve the location of the zeros in the sector $|\operatorname{Arg} z| > \frac{3\pi}{4}$ are of interest, in part because a real entire function whose zeros lie in this open sector is automatically log-concave (cf. Theorem 71).

The bound $\alpha^4 \geq 2$ on the Turán constant required to apply Theorem 81 is smaller than that in (4.9) and Theorem 83, and indeed, an entire function whose coefficients satisfy the Turán inequalities with Turán constant $\sqrt{2} < \alpha < 2$ does not necessarily belong to the Laguerre-Pólya class, yet operating with S_r will produce a function in the Laguerre-Pólya class.

Example 85. $f(x) = \sum_{k=0}^{\infty} 2^{-\frac{k^2}{2}} x^k \notin \mathscr{L}\text{-}\mathscr{P}$, because the third Jensen polynomial associated to $f(x)$ has a pair of non-real zeros. However, the coefficients of $f(x)$ are decreasing and satisfy $a_k^2 - 2a_{k-1}a_{k+1} \geq 0$ for all positive integers k. By Theorem 81, for all positive integers r, the coefficients of $S_r[f(x)]$ satisfy the Turán inequalities with Turán constant 4, and whence by Theorem 75, $S_r[f(x)] \in \mathscr{L}\text{-}\mathscr{P}^+$ for all positive integers r. □

A converse to Corollary 82 would be particularly pleasant, however it is not the case that if $S_r[f(x)] \in \mathscr{L}\text{-}\mathscr{P}^+$ for all positive integers r, then $f(x) \in \mathscr{L}\text{-}\mathscr{P}^+$, as the following example illustrates.

Example 86. Consider the polynomial $p(x) = (x+1)^3 \in \mathscr{L}\text{-}\mathscr{P}^+$. By Theorem 42, $S_1[p(x)] \in \mathscr{L}\text{-}\mathscr{P}^+$ and for $r \geq 2, S_r[p(x)] = \Lambda^0(z^2+1)[p(x)] \in \mathscr{L}\text{-}\mathscr{P}^+$. However, $a_1^2 - 4a_0a_2 = -3$; i.e., the first Turán inequality fails. □

By controlling the decay of the rapidly decreasing sequences we investigate a partial converse to Theorem 81 and obtain sufficient conditions for a sequence to be log-concave in terms of the non-linear operator $S_r : a_k \mapsto a_k^2 - a_{k-r}a_{k+r}$.

Theorem 87. *Let $\{a_k\}_{k=0}^{\omega}, 0 \leq \omega \leq \infty$, be a real sequence with terms that alternate in sign or have the same sign. Suppose that for some positive integer r and some $\alpha > 0$, the sequence $\{b_{k,r} := a_k^2 - \alpha^2 a_{k-r}a_{k+r}\}_{k=0}^{\infty}$ satisfies the Turán-type inequalities $b_{k,r}^2 - \alpha^2 b_{k-1,r}b_{k+1,r} \geq 0$. Then, for each positive integer k, the sequence $\{a_k\}_{k=0}^{\infty}$ satisfies $a_k^2 - \alpha a_{k-1}a_{k+1} \geq 0$, provided that for some $\beta \geq \alpha$ and all positive integers k,*

(i) *(a)* $|a_{k-1}| \geq |a_k|$, *and*

(b) $\beta|a_k| \geq \alpha|a_{k-1}|$, *and*

(ii) *(a)* $\frac{\alpha^2}{2} + \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^r - \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^2 \geq \left(\frac{\beta}{\alpha}\right)^r$, *or*

(b) $a_k^2 = \frac{\alpha^2}{2}a_{k-r}a_{k+r}$ *and* $\frac{\alpha^2}{2} \geq \left(\frac{\beta}{\alpha}\right)^r$.

Proof. Fix a positive integer r. The sequence $\{b_{k,r}\}_{k=0}^{\infty}$ satisfies the Turán inequalities with Turán constant α^2. Expanding $b_{k,r}^2 - \alpha^2 b_{k-1,r}b_{k+1,r} \geq 0$, we obtain

$$\begin{aligned}
a_k^4 - 2\alpha^2 a_k^2 a_{k-r}a_{k+r} + \alpha^4 a_{k-r}^2 a_{k+r}^2 &\geq \alpha^2 a_{k-1}^2 a_{k+1}^2 - \alpha^4 a_{k-1}^2 a_{k-r+1}a_{k+r+1} \\
&\quad - \alpha^4 a_{k+1}^2 a_{k-r-1}a_{k+r-1} \\
&\quad + \alpha^6 a_{k-r-1}a_{k+r-1}a_{k-r+1}a_{k+r+1} \\
a_k^4 - \alpha^2 a_{k-1}^2 a_{k+1}^2 &\geq 2\alpha^2 a_k^2 a_{k-r}a_{k+r} - \alpha^4 a_{k-1}^2 a_{k-r+1}a_{k+r+1} \\
&\quad - \alpha^4 a_{k-r}^2 a_{k+r}^2 - \alpha^4 a_{k+1}^2 a_{k-r-1}a_{k+r-1} \\
&\quad + \alpha^6 a_{k-r-1}a_{k+r-1}a_{k-r+1}a_{k+r+1} \; .
\end{aligned} \tag{4.20}$$

Observe that products of the form $a_{k-m}a_{k+m}, m = 0, 1, 2, \ldots$, are positive. We will verify that under each set of hypotheses $a_k^4 - \alpha^2 a_{k-1}^2 a_{k+1}^2 \geq 0, k = 1, 2, 3 \ldots$, which implies the log-concavity of the sequence $\{a_k\}_{k=0}^{\infty}$. We may assume that $k - r + 1 \geq 0$, else according to our conventions $a_{k-r+1} := 0$ (cf. Notation 36), and there is nothing to prove. Indeed, if $k - r + 1 < 0$, then also $k - r < 0$ and $k - r - 1 < 0$ so that (4.20) becomes $a_k^4 - \alpha^2 a_{k-1}^2 a_{k+1}^2 \geq 0$.

First, iterating i(b), we obtain

$$\frac{\beta}{\alpha}|a_k| \leq \left(\frac{\beta}{\alpha}\right)^2 |a_{k+1}| \cdots \leq \left(\frac{\beta}{\alpha}\right)^r |a_{k+r-1}| \leq \left(\frac{\beta}{\alpha}\right)^{r+1} |a_{k+r}| \leq \left(\frac{\beta}{\alpha}\right)^{r+2} |a_{k+r+1}| \, . \tag{4.21}$$

Next, shifting indices in i(a) by $k \mapsto k+r$ and and i(b) by $k \mapsto k+r+1$, we obtain $|a_{k+r-1}| \geq |a_{k+r}|$ and $\frac{\beta}{\alpha}|a_{k+r+1}| \geq |a_{k+r}|$. Thus, an upper bound of a_{k+r}^2 is $\frac{\beta}{\alpha}a_{k+r-1}a_{k+r+1}$. Similarly, shifting

shifting indices in i(a) by $k \mapsto k-r$ and in i(b) by $k \mapsto k-r+1$, we obtain $|a_{k-r}| \leq |a_{k-r-1}|$ and $|a_{k-r}| \leq \frac{\beta}{\alpha}|a_{k-r+1}|$. Thus, an upper bound for a_{k-r}^2 is $\frac{\sqrt{2}}{\alpha} a_{k-r-1}a_{k-r+1}$, and whence

$$\alpha^4 a_{k-r}^2 a_{k+r}^2 \leq \beta^2\alpha^2 a_{k-r-1}a_{k+r-1}a_{k-r+1}a_{k+r+1}. \tag{4.22}$$

Next, $|a_{k+1}| \leq |a_{k-r+1}|$ and, by (4.21), $\left(\frac{\beta}{\alpha}\right)^2 |a_{k+1}| \leq \left(\frac{\beta}{\alpha}\right)^{r+2} |a_{k+r+1}|$. Thus, an upper bound for a_{k+1}^2 is $\left(\frac{\beta}{\alpha}\right)^r a_{k-r+1}a_{k+r+1}$, and

$$\alpha^4 a_{k+1}^2 (a_{k-r-1}a_{k+r-1}) \leq \alpha^4 \left(\frac{\beta}{\alpha}\right)^r (a_{k-r-1}a_{k+r-1})a_{k-r+1}a_{k+r+1}. \tag{4.23}$$

Similarly, $|a_k| \leq |a_{k-r}|$ and from (4.21) we obtain $\left(\frac{\beta}{\alpha}\right)|a_k| \leq \left(\frac{\beta}{\alpha}\right)^{r+1} |a_{k+r}|$. Thus, after shifting the index by $k \mapsto k-1$, an upper bound for a_{k-1}^2 is $\left(\frac{\beta}{\alpha}\right)^r a_{k+r-1}a_{k-r-1}$, and

$$\alpha^4 a_{k-1}^2 (a_{k-r+1}a_{k+r+1}) \leq \alpha^4 \left(\frac{\beta}{\alpha}\right)^r a_{k-r-1}a_{k+r-1}(a_{k-r+1}a_{k+r+1}). \tag{4.24}$$

Next, shifting indices in i(a) by $k \mapsto k-r+1$ and in i(b) by $k \mapsto k+r$ we have $|a_{k-r}| \geq |a_{k-r+1}|$ and $|a_{k+r}| \geq \frac{\alpha}{\beta}|a_{k+r-1}|$, so a lower bound for $a_{k-r}a_{k+r}$ is $\frac{\alpha}{\sqrt{2}} a_{k-r+1}a_{k+r-1}$. Now, $|a_{k-r-1}| \geq |a_k|$ and shifting the index by $k \mapsto k+r+1$, we have the lower bound $|a_k| \geq \beta^{r+1}|a_{k+r+1}|$. Also, from (4.21) we obtain $\left(\frac{\beta}{\alpha}\right)^{r+2} |a_{k+r+1}| \geq \left(\frac{\beta}{\alpha}\right)|a_k|$ and shifting the index by $k \mapsto k-r-1$, we have the lower bound $|a_k| \geq \left(\frac{\alpha}{\beta}\right)^{r+1} |a_{k-r-1}|$. Thus, a lower bound for a_k^2 is $\left(\frac{\alpha}{\beta}\right)^{r+1} a_{k+r+1}a_{k-r-1}$, and whence

$$2\alpha^2 a_k^2 a_{k-r}a_{k+r} \geq \alpha^4 \left(\frac{\alpha}{\beta}\right)^r a_{k-r-1}a_{k+r-1}a_{k-r+1}a_{k+r+1}. \tag{4.25}$$

Finally, to ensure that the right member of (4.20) is non-negative we estimate it using (4.22), (4.23), (4.24), (4.25), and obtain the lower bound

$$a_k^4 - \alpha^2 a_{k-1}^2 a_{k+1}^2 \geq a_{k-r-1}a_{k+r-1}a_{k-r+1}a_{k+r+1}\left(\alpha^4\left(\frac{\alpha}{\beta}\right)^r - 2\alpha^4\left(\frac{\beta}{\alpha}\right)^r - \beta^2\alpha^2 + \alpha^6\right). \tag{4.26}$$

The right member of (4.26) is non-negative provided $\alpha^6 + \alpha^4\left(\frac{\alpha}{\beta}\right)^r \geq 2\alpha^4\left(\frac{\beta}{\alpha}\right)^r + \beta^2\alpha^2$. Simplified, this is condition (iia).

To verify ii(b), we proceed in a similar way, but consider first the inequality

$$2\alpha^2 a_k^2 a_{k-r} a_{k+r} - \alpha^4 a_{k-r}^2 a_{k+r}^2 \geq 0 \, , \tag{4.27}$$

which holds, provided that $\alpha_k^2 \geq \frac{\alpha^2}{2} a_{k-r} a_{k+r}$. For the remaining terms, we will show that

$$\begin{aligned} &\frac{\alpha^6}{2} a_{k-r-1} a_{k+r-1} a_{k-r+1} a_{k+r+1} - \alpha^4 a_{k+1}^2 a_{k-r-1} a_{k+r-1} \geq 0, \text{and} \\ &\frac{\alpha^6}{2} a_{k-r-1} a_{k+r-1} a_{k-r+1} a_{k+r+1} - \alpha^4 a_{k-1}^2 a_{k-r+1} a_{k+r+1} \geq 0 \, . \end{aligned} \tag{4.28}$$

The first inequality in (4.28) is, by ii(a), equivalent to

$$\alpha^2 a_{k-r-1} a_{k+r-1} \left(\frac{\alpha^2}{2} a_{k-r+1} a_{k+r+1} - a_{k+1}^2 \right) \geq 0 \, , \tag{4.29}$$

and the second inequality in (4.28) reduces to

$$\alpha^2 a_{k-r+1} a_{k+r+1} \left(\frac{\alpha^2}{2} a_{k+r-1} a_{k-r-1} - a_{k-1}^2 \right) \geq 0 \, . \tag{4.30}$$

The expressions in parentheses must both be positive and, taken together, if follows from (4.23) and (4.24) that inequalities (4.29) and (4.30) hold, provided

$$\frac{\alpha^2}{2} \geq \left(\frac{\beta}{\alpha} \right)^r \, . \tag{4.31}$$

□

When $r = 1$ we see that inequality in ii(b) becomes $\alpha \leq \beta \leq \frac{\alpha^3}{2}$, and as r grows without bound we see that α tends to β and the sequence $\{a_k\}_{k=0}^{\omega}$ becomes a constant sequence via condition (i).

The sequence $\{b_{k,r}\}_{k=0}^{\omega}$ need not be non-negative; however, if $\{a_k\}_{k=0}^{\omega}$ is generated by a function in $\mathscr{L}$-$\mathscr{P}^+$ and $\alpha = 1$, then by the total positivity of $\{a_k\}_{k=0}^{\omega}$ the sequence $\{b_{k,r}\}_{k=0}^{\infty}$ is in fact a positive sequence. On the other hand, choosing $\alpha = 1$ and letting $\{a_k\}_{k=0}^{\omega}$ be generated by a function in $\mathscr{L}$-$\mathscr{P}^+$ does not satisfy neither of the remaining hypotheses, whence Theorem 87 is not a theorem about the Laguerre-Pólya class.

The difficulty in applying Theorem 87 lies in verifying the hypothesis that the sequence $\{a_k^2 - \alpha^2 a_{k-r} a_{k+r}\}_{k=0}^{\omega}$ is α^2-log-concave. We know that the non-linear operator $S_r : a_k \mapsto a_k^2 - a_{k-r} a_{k+r}$

does not in general preserve $\mathscr{L}$-$\mathscr{P}^+$ with the notable exceptions of $r = 1, 2, 3, 4$, and introducing the parameter α^2 does not remedy this situation.

If we assume that $a_k^2 = \frac{\alpha^2}{2} a_{k-r} a_{k+r}$, then $b_{k,r} = -\frac{\alpha^2}{2} a_{k-r} a_{k+r}$, which satisfies i(a), trivially, and from i(b) we see that the inequalities $a_k^2 \geq \frac{\alpha}{\beta} a_{k-1} a_{k+1}$ must hold for each positive integer k. Hence, we must have $\frac{\alpha^2}{2} \geq \frac{\alpha}{\beta}$, or $\alpha\beta \geq 2$.

We pass to some special choices of α we have encountered in the prequel. In light of Theorem 78, we choose $\alpha = x_0$, the positive zero of $x^3 - x^2 - 2x - 1$.

Corollary 88. *Let $\{a_k\}_{k=0}^{\omega}, 0 \leq \omega \leq \infty$, be a sequence of positive numbers that is non-increasing and satisfies, for each positive integer k and some $\beta \geq \sqrt{x_0} \approx 1.4656$, the inequalities $\beta a_k \geq \sqrt{x_0} a_{k-1}$, where $x_0 \approx 2.1479$ is the positive zero of $x^3 - x^2 - 2x - 1$. If, for some positive integer r, the sequence $\{a_k^2 - x_0 a_{k-r} a_{k+r}\}_{k=0}^{\infty}$ is x_0-log-concave, then the function $f(x) = \sum_{k=0}^{\omega} a_k x^k$ that generates the sequence $\{a_k\}_{k=0}^{\omega}$ is Hurwitz stable, provided that*

(i) $r = 1$ *and* $\beta \leq q_1 \approx 3.2337$, *or*

(ii) $r = 2$ *and* $\beta \leq q_2 \approx 2.8255$, *or*

(iii) $r = 3$ *and* $\beta \leq q_3 \approx 2.6279$, *or*

(iv) $r = 4$ *and* $\beta \leq q_4 \approx 2.5169$.

Proof. It suffices to verify condition ii(a) of Theorem 87. The numbers $q_r, r = 1, 2, 3, 4$, are the positive roots of the numerator of the expression in ii(a) with $\alpha = \sqrt{x_0}$, and $r = 1, 2, 3, 4$, respectively, obtained with the help of Mathematica.

Having established the $\sqrt{x_0}$-log-concavity of the sequence $\{a_k\}_{k=0}^{\omega}$, an application of Theorem 78 allows us to conclude that any polynomial generating the sequence $\{a_k\}_{k=0}^{\omega}$ is Hurwitz stable. By Corollary 80, functions generating the sequence $\{a_k\}_{k=0}^{\infty}$ are entire and thus our conclusion also holds when $\omega = \infty$. □

In light of Theorem 83, we choose $\alpha = \frac{3+\sqrt{5}}{2}$.

Corollary 89. *Let $\{a_k\}_{k=0}^{\omega}, 0 \leq \omega \leq \infty$, be a sequence of positive numbers that is non-increasing and satisfies, for each positive integer k and some $\beta > \frac{3+\sqrt{5}}{2}$, the inequalities $\beta a_k \geq \frac{3+\sqrt{5}}{2} a_{k-1}$. If for some positive integer r the sequence $\{a_k^2 - \frac{7+3\sqrt{5}}{2} a_{k-r} a_{k+r}\}_{k=0}^{\infty}$ is $\left(\frac{7+3\sqrt{5}}{2}\right)$-log-concave, then the sequence $\{a_k\}_{k=0}^{\infty}$ is infinitely log concave, provided* *(i)* $r = 1$ *and* $\beta \leq t_1 \approx 4.9615$*, or*

(ii) $r = 2$ *and* $\beta \leq t_2 \approx 4.0747$*, or*

(iii) $r = 3$ *and* $\beta \leq t_3 \approx 3.6258$*, or*

(iv) $r = 4$ *and* $\beta \leq t_4 \approx 3.3789$*.*

Proof. It suffices to verify condition ii(a) of Theorem 87. The numbers $t_r, r = 1, 2, 3, 4$, are the positive roots of the numerator of the expression in ii(a) with $\alpha = \frac{3+\sqrt{5}}{2}$, and $r = 1, 2, 3, 4$, respectively, obtained with the help of Mathematica. $\square$

Despite the difficult calculations necessary to establish the hypotheses Corollaries 88 and 89, one redeeming consequence is that these result will hold, *mutatis mutandis*, for values of α larger than those specified in the hypotheses.

Next, we consider an example of a sequence of positive numbers satisfying ii(b).

Example 90. Let $\alpha \geq 1$ and consider the sequence $\{a_k\}_{k=0}^{\infty}$, where $a_0 = 1, a_1 = \frac{1}{\alpha}$, and for $k \geq 2$ we have the recurrence relation $a_{k+1} = \frac{2a_k^2}{\alpha^2 a_{k-1}}$. Then, the first several terms are given by

$$\left\{1, \frac{1}{\alpha}, \frac{2}{\alpha^4}, \frac{2^3}{\alpha^9}, \frac{2^6}{\alpha^{16}}, \frac{2^{10}}{\alpha^{25}}, \ldots, \frac{2^{\frac{k(k+1)}{2}}}{\alpha^{k^2}}, \ldots\right\}_{k=0}^{\infty}. \tag{4.32}$$

We will also assume that $\beta = \alpha^2$, so that ii(b) holds. To verify i(a), we shift the index, and obtain

$$\frac{2^{\frac{k(k-1)}{2}}}{\alpha^{k^2}} \geq \frac{2^{\frac{k(k+1)}{2}}}{\alpha^{(k+1)^2}}, \tag{4.33}$$

which holds provided $\alpha^{2k+1} \geq 2^k$, or $\alpha \leq \sqrt[3]{2}$. Next, with the assumption that $\beta = \alpha^2$, we consider i(b) and obtain

$$\alpha \cdot \frac{2^{\frac{k(k+1)}{2}}}{\alpha^{(k+1)^2}} \geq \frac{2^{\frac{k(k-1)}{2}}}{\alpha^{k^2}}, \tag{4.34}$$

which holds provided $2^k \geq \alpha^k$. Thus, conditions (i) and ii(b) hold for $\sqrt[3]{2} \leq \alpha \leq 2$ and (4.32). It remains to check that the sequence $\{b_{k,1} := a_k^2 - \alpha^2 a_{k-1}a_{k+1}\}_{k=0}^{\infty}$ satisfies the inequalities $b_{k,1}^2 - \alpha^2 b_{k-1,1}b_{k+1,1} \geq 0$ for all positive integers k. A calculation shows that $b_{k,1} = -\frac{2^{k(k-1)}}{\alpha^{2k^2}}$, whence $b_{k,1}^2 - \alpha^2 b_{k-1,1}b_{k+1,1} = (\alpha^2 - 4)\frac{4^{k(k-1)}}{\alpha^{4k^2+2}}$. This forces $\alpha = 2$ and consequently (4.32) becomes

$$\left\{1, \frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^6}, \frac{1}{2^{10}}, \frac{1}{2^{15}}, \ldots, \frac{1}{2^{\frac{k(k-1)}{2}}}, \ldots\right\}_{k=1}^{\infty}, \tag{4.35}$$

which is a 2-log-concave sequence. $\square$

4.1.4 A question of S. Fisk.

The next result is, in part, motivated by the following problem of S. Fisk. We recall that the Bell polynomial of degree n is given by the generating function

$$e^x B_n(x) = \sum_{k=0}^{\infty} \frac{k^n}{k!} x^k, \tag{4.36}$$

and it is also known that $B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$, where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind, the number of ways a non-empty set of n elements can be partitioned into k non-empty subsets.

Problem 91 ([37, # 32, p. 732])**.** *Let $B_n(x)$ denote the Bell polynomial of degree n. Is it true that for each positive integer n the polynomial $B_n(x)B_{n+2}(x) - (B_{n+1}(x))^2$ is (weakly) Hurwitz stable?*

Theorem 92. *Let $\sum_{k=0}^{\infty} a_k x^k$ be a real entire function with positive coefficients and denote by $p_n(x)$ its partial sum $\sum_{k=0}^{n} a_k x^k$. Suppose that the sequence $\{a_k\}_{k=0}^{\infty}$ is A-log-concave and that the sequence $\{\frac{a_k}{a_{k+1}}\}_{k=0}^{\infty}$ is non-decreasing. Then, for each positive integer n, coefficient sequences generated by the polynomials*

$$p_{n+1}^2(x) - p_n(x)p_{n+2}(x) = x^n \sum_{k=0}^{n+1} (a_k a_{n+1} - a_{k-1}a_{n+2}) x^k = x^n \sum_{k=0}^{n+1} \beta_k x^k, \tag{4.37}$$

satisfy $\beta_k^2 - A\beta_{k-1}\beta_{k+1} \geq 0$, provided $A \geq 2$.

Proof. A direct calculation yields,

$$\begin{aligned} \frac{\beta_{k+1}^2 - A\beta_{k-1}\beta_{k+2}}{a_{n+2}^2} &= \frac{(a_k a_{n+1} - a_{k-1}a_{n+2})^2 - A(a_{k-1}a_{n+1} - a_{k-2}a_{n+2})(a_{k+1}a_{n+1} - a_k a_{n+2})}{a_{n+2}^2} \\ &= \left(\frac{a_{n+1}}{a_{n+2}}\right)^2 (a_k^2 - Aa_{k-1}a_{k+1}) + \left(\frac{a_{n+1}}{a_{n+2}}\right)(A-2)a_k a_{k-1} \\ &\quad + \left(\frac{a_{n+1}}{a_{n+2}}\right) A a_{k+1}a_{k-2} - A a_k a_{k-2} + a_{k-1}^2, \end{aligned} \tag{4.38}$$

where the first two terms are non-negative, because we have assumed the sequence $\{a_k\}_{k=0}^{\infty}$ to be

A-log-concave with $A \geq 2$. It remains to be shown that last term is non-negative, but

$$\left(\frac{a_{n+1}}{a_{n+2}}\right) A a_{k+1} a_{k-2} - A a_k a_{k-2} + a_{k-1}^2 \geq A a_{k-2} \left(\left(\frac{a_{n+1}}{a_{n+2}}\right) a_{k+1} - a_k \right), \tag{4.39}$$

and $\left(\frac{a_{n+1}}{a_{n+2}}\right) \geq \left(\frac{a_k}{a_{k+1}}\right)$ follows from the assumption that the sequence $\{\frac{a_k}{a_{k+1}}\}_{k=0}^{\infty}$ is non-decreasing.

□

In light of Hutchinson's Theorem, we have the following corollary regarding the partial sums of real entire functions that generate rapidly decreasing sequences.

Corollary 93. *Let $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k$ be a real entire function with positive coefficients and denote by $p_n(x)$ its partial sum $\sum_{k=0}^{n} a_k x^k$. If $\varphi(x)$ generates a rapidly decreasing sequence, then $p_{n+1}^2(x) - p_n(x) p_{n+2}(x) \in \mathscr{L}\text{-}\mathscr{P}^+$.* □

Example 94. The coefficients of the Bell polynomials, the Stirling numbers of the second kind, are not 2-log-concave. With the help of Mathematica, we see that the sequence

$$\left\{ \left(\left\{ \begin{matrix} 11 \\ k \end{matrix} \right\} \right)^2 - 2 \left\{ \begin{matrix} 11 \\ k-1 \end{matrix} \right\} \left\{ \begin{matrix} 11 \\ k+1 \end{matrix} \right\} \right\}_{k=1}^{10} \tag{4.40}$$

contains negative terms. □

In spite of the above example, we show in Proposition 111 that the Stirling numbers of the second kind, like the binomial coefficients, are in fact infinitely log-concave.

4.1.5 Coefficient inequalities and strip properties

Motivated by the following elementary observation and a classical problem, real and highly-concave polynomials, all of whose zeros lie in a strip, are of primary interest in this section. Consider the open strip $S(\lambda) := \{z \in \mathbb{C} \mid |\Im z| < \lambda\}$ and the class of real polynomials all of whose zeros are contained in $S(\lambda)$. We may regard a real polynomial all of whose zeros are non-real and lie in $S(\lambda)$, for some $\lambda > 0$, as a product of irreducible quadratic polynomials $q_k(z)$. Let $\alpha_k = a_k + b_k i$ be a zero of $q_k(z)$. Then, $q_k(z) = (z - \alpha_k)(z - \overline{\alpha_k}) = z^2 - 2a_k z + a_k^2 + b_k^2$, and the lone Turán ratio of $q_k(z)$ is

$$\beta_k = \frac{4a_k^2}{a_k^2 + b_k^2} \geq \frac{4a_k^2}{a_k^2 + \lambda^2}. \tag{4.41}$$

In order for the zeros of $q_k(z)$ to lie in the strip $S(\lambda)$, it is necessary, by Hutchinson's Theorem, to impose on (4.41) the restriction $0 < \lambda^2 < 3a_k^2$. Moreover, with the above lower bound on β_k, we have

$$\beta_k(a_k^2 + \lambda^2) \geq 4a_k^2 \iff \lambda^2 \geq (4 - \beta_k)\frac{a_k^2}{\beta_k} > 0, \tag{4.42}$$

and combining (4.41) and (4.42),

$$\begin{aligned} 3a_k^2 > \lambda^2 \geq (4 - \beta_k)\frac{a_k^2}{\beta_k} > 0 &\iff \sqrt{3} > \sqrt{\frac{4 - \beta_k}{\beta_k}} > 0 \\ &\iff \beta_k > 1. \end{aligned} \tag{4.43}$$

Problem 95. *Let* $\varphi(x) = \sum_{k=0}^{\omega} a_k x^k, 2 \leq \omega \leq \infty$, *be a real entire function and let* $\beta_\varphi = \{\beta_k := \frac{a_k^2}{a_{k-1}a_{k+1}}\}_{k=1}^{\omega-1}$ *be the sequence of its Turán ratios. If* $\varphi(x) = \sum_{k=0}^{N} a_k x^k + \sum_{k=N+1}^{\omega} a_k x^k$, *where the zeros of the polynomial* $\sum_{k=0}^{N} a_k x^k$ *all lie in some open strip* $S(\lambda)$, *then what conditions on the sequence* β_φ *are necessary and/or sufficient to guarantee that the zeros of* $\varphi(x)$ *all lie in* $S(\lambda)$*?*

We follow the method of D. Handelmann ([46]) and obtain partial results toward answering the above question. We will need a refined notion of the Turán constant for sequences of positive numbers (cf. Definition 74) and some classical results, beginning with a theorem of A. Hurwitz.

Definition 96. From a real entire function $\varphi(x)$ generating the sequence $\{a_k\}_{k=0}^{\omega}, 1 \leq \omega \leq \infty$, all of whose terms are positive, we obtain the associated sequence of ratios $\left\{\beta_k := \frac{a_k^2}{a_{k-1}a_{k+1}}\right\}_{k=1}^{\omega-1}$. Let $\beta_{\varphi,G}$ be the infimum over all such sequences for which the associated family of real entire functions has all its zeros in an open connected subset G of $\mathbb{C}$. □

Theorem 97 ((A. Hurwitz) [16, p. 152])**.** *Let* $\{f_k\}_{k=0}^{\infty}$ *be a sequence of functions analytic in an open connected subset* G *of* $\mathbb{C}$ *that converges, on compact subsets of* G, *to a non-constant function* f. *If* $f(z_0) = 0$ *for some* $z_0 \in G$, *then for all* $\epsilon > 0$ *there exists and integer* N, *sufficiently large, such that for all* $n \geq N$ *the functions* f *and* f_n *have the same number of zeros, counting multiplicities, in an open disk of radius* ϵ *centered at* z_0. □

In conjunction with Hurwitz's Theorem, the next results state that the zeros of an analytic function vary continuously with its coefficients.

Lemma 98 ([79, p. 20]). *Let* $f(z) = \sum_{k=0}^{n} a_n z_k \in \mathbb{C}[z]$ *and let* $\{f_j\}_{j=0}^{\infty}$ *be a sequence of polynomials, such that* $f_j(z) = \sum_{k=0}^{n} a_{k,j} z^k$ *for each* j*, and* $\lim_{j\to\infty} a_{k,j} = a_k$*, for each* k*. Then,* $\{f_j\}_{j=0}^{\infty}$ *converges uniformly, on compact subsets of* $\mathbb{C}$*, to* f*.*

Proof. Let G be a compact subset of $\mathbb{C}$ and let $\epsilon > 0$. Since $\lim_{k\to\infty} a_{k,j} = a_k$ for each k, there exists a positive integer N_k such that for all $j \geq N_k$, $|a_k - a_{k,j}| < \frac{\epsilon}{(n+1)M^n}$, where $M^n = \max\{1, \sup_{z\in G}|z|\}$. Thus, for $j \geq \max_k\{N_k\}$ and $z \in G$, we have

$$\begin{aligned} |f(z) - f_j(z)| &= \left|\sum_{k=0}^{n}(a_k - a_{k,j})z^k\right| \\ &\leq \sum_{k=0}^{n} |a_k - a_{k,j}|\,|z|^k \\ &< \sum_{k=0}^{n} \frac{\epsilon}{(n+1)M^n} M^n \\ &= \epsilon. \end{aligned} \tag{4.44}$$

□

Lemma 99. *Let* $G \subset \mathbb{C}$ *be a non-empty, open connected set and let* P_N *denote the collection of all real polynomials of degree at most* N*. If none of the elements of* P_N *have zeros on the boundary of* G*, then the function* $u : P_N \to \mathbb{N}$*, that counts the number of zeros of* $p \in P_N$ *in* G*, is continuous.*

Proof. Let $p(z) = \sum_{k=0}^{m} a_k z^k, m \leq N$, and $\tilde{p}(z) = \sum_{k=0}^{n} b_k z^k, n \leq N$, be two polynomials in P_N. Let $\epsilon > 0$, and for each k suppose that $|a_k - b_k| \leq \delta = \frac{\epsilon}{(N+1)\max\{m,n\}}$. Then,

$$\begin{aligned} |p(z) - \tilde{p}(z)| &= \sum_{k=0}^{\max\{m,n\}} |a_k - b_k| \\ &\leq \sum_{k=0}^{\max\{m,n\}} \frac{\epsilon}{(N+1)\max\{m,n\}} \\ &= \frac{\epsilon}{(N+1)} < \epsilon. \end{aligned} \tag{4.45}$$

Now, let $p(z)$ by approximated by the sequence $\{p_j(z)\}_{j=0}^{\infty}$, where $p_j(z) = \sum_{k=0}^{m} a_{k,j} z^k$, and $a_k = \lim_{j\to\infty} a_{k,j}$, for each k. By Lemma 98, $\{p_j(z)\}_{j=0}^{\infty}$ approximates $p(z)$ locally uniformly. Let $z_0 \in G$ be a zero of $p(z)$ and let $D(z_0, \epsilon)$ be an open disk of radius ϵ centered at z_0. Then, there exists an integer N sufficiently large such that for all $j \geq N$, (4.45) holds for $p(z)$ and $p_j(z)$ on $D(z_0, \epsilon)$. Let $\tilde{p}(z) = p_j(z)$, for some $j \geq N$. By Hurwitz's Theorem, both $p(z)$ and $\tilde{p}(z)$ have the same number of zeros, counting multiplicities, in $D(z_0, \epsilon)$. Thus, $u(p(z)) = u(\tilde{p}(z))$ on $D(z_0, \epsilon)$, and whence by the Uniqueness Theorem (see for example [68, §82]), on G. $\square$

The next result may be regarded as a supplement to Lemma 98.

Lemma 100. *Let $U \subset P_N$ and $G \subset \mathbb{C}$ be non-empty, open and connected sets. Suppose that the elements of U are all of the same degree and none of them have a zero on the boundary of G. If there exists an element $p \in U$ all of whose zeros lie in G, then all zeros of all elements of U lie in G.*

Proof. Fix a positive integer N and let $U \subset P_N$ be an open connected set. Let $u : U \to \mathbb{N}$ be the function that given an element $p \in U$ counts the number of zeros of p in G. By Lemma 99, the function u is continuous, and whence

$$U = \bigcup_{n=0}^{N} u^{-1}(n), \tag{4.46}$$

where, by the continuity of u, each $u^{-1}(n) \subset U$ is open. By hypothesis, there exists a $p \in U$, say of degree $n_0 \leq N$, with all its zeros contained in G, whence U is non-empty, and because U is assumed to be connected we must have $U = u^{-1}(n_0)$. $\square$

Next, we construct a specific class of real polynomials all of whose Turán ratios (cf. Definition 96) are the same.

Proposition 101. *Let $N \geq 2$ be a positive integer and let $n \in \mathbb{R} \setminus \{0\}$. There exists a real monic polynomial $p_{n,N}$ of degree N all of whose Turán ratios are equal to $\frac{1}{n^2}$.*

Proof. We construct the desired polynomials by setting $p_{n,N}(x) = \sum_{k=0}^{N} \frac{z^k}{n^{a_{k,N}}}$. For each positive integer $N \geq 2$, the sequence $\{a_{k,N}\}_{k=0}^{N}$ is defined to be the $(N+1)^{th}$ row of the triangle obtained from the upper left corner of the infinite matrix $(a_{i,j})$, where $a_{i,j} = ij$,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 2 & 4 & 6 & 8 & 10 & \cdots \\ 0 & 3 & 6 & 9 & 12 & 15 & \cdots \\ 0 & 4 & 8 & 12 & 16 & 20 & \cdots \\ 0 & 5 & 10 & 15 & 20 & 25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.47}$$

Then, the sequence of Turán ratios is given by

$$\left\{ \frac{n^{a_{k-1,N}} n^{a_{k+1,N}}}{\left(n^{a_{k,N}}\right)^2} \right\}_{k=1}^{N-1} = \left\{ n^{a_{k-1,N}+a_{k+1,N}-2a_{k,N}} \right\}_{k=1}^{N-1}. \tag{4.48}$$

Observe that $a_{k,N} = k(N-k), k = 0, 1, \dots, N$, and whence $a_{k,N+1} = k + a_{k,N}, N \geq 2$. Initially, when $N = 2$, we have $\{a_{k-1,2} + a_{k+1,2} - 2a_{k,2}\}_{k=1}^{1} = \{-2\}$. We assume that $a_{k-1,N} + a_{k+1,N} - 2a_{k,N} = -2$, for a fixed $N \geq 2$ and each $k = 1, \dots, N-1$. Induction on N, yields

$$\begin{aligned} a_{k-1,N+1} + a_{k+1,N+1} - 2a_{k,N+1} &= a_{k-1,N} + (k-1) + a_{k+1,N} + (k+1) - 2(a_{k,N} + k) \\ &= a_{k-1,N} + a_{k+1,N} - 2a_{k,N} \\ &= -2. \end{aligned} \tag{4.49}$$

Next, under the same induction hypothesis, we induct on k. For $N \geq 2$, and a fixed $k = 1, 2, \dots, N-1$, we observe that $a_{k+1,N} = a_{k,N} + N - 2k - 1$. Then, for $k = 0, 1, \dots, N-2$, we have

$$\begin{aligned} a_{k,N} + a_{k+2,N} - 2a_{k+1,N} &= a_{k-1,N} + (N - 2k + 1) + a_{k+1,N} + N - 2k - 3 - 2(a_{k,N} + N - 2k - 1) \\ &= a_{k-1,N} + a_{k+1,N} - 2a_{k,N} \\ &= -2. \end{aligned} \tag{4.50}$$

□

Lemma 102. *Let g_k be a real highly-concave polynomial of degree $k \geq 2$, all of whose Turán ratios are equal, and let $p_{n,N-k}$ be a real polynomial of degree $N - \deg g_k \geq 2$, chosen as in Proposition 101, so that all of its Turán ratios are equal to $\frac{1}{n^2}$, for some $n \in \mathbb{R} \setminus \{0\}$. Then, the Turán constant of the polynomial $p_{n,N-k}g_k$, as $n \to \infty$, is at least as large as the Turán constant of g_k.*

Proof. Let $g_k(x) = \sum_{j=0}^{k} a_j x^j, k \geq 2$, and let $p_{n,N-k}(x) = \sum_{k=0}^{N-k} b_k x^k, N \geq 4$. The product $p_{n,N-k}g_k(x)$ is a polynomial of degree $N \geq 2$ whose coefficients are given by the Cauchy product $\sum_{r=0}^{N} \sum_{j+k=r} a_j b_{k-j} x^r$, and its m^{th} Turán ratio, $m = 1, 2, \ldots, N-1$, is given by

$$\frac{(a_0 b_m + \cdots + a_m b_0)^2}{(a_0 b_{m-1} + \cdots + b_0 a_{m-1})(a_0 b_{m+1} + \cdots + a_{m+1} b_0)}, \tag{4.51}$$

where $a_0 = 1$, and for $s = 1, 2, \ldots, m, a_s = \frac{1}{n^p}, p \geq 1$. Thus, as $n \to \infty$, the expression (4.51) is at least as large as $\frac{b_m^2}{b_{m-1}b_{m+1}}$, the m^{th} Turán ratio of g_k, and whence the same is true for the smallest such ratios. $\square$

Lemma 103. *For real highly log-concave polynomials p of degree at least 5 and $0 < \lambda < 1$,*

$$\beta_{p,S(\lambda)} \geq \max\left\{1, \sqrt{2}\lambda, 4(1-\lambda^2), 3\sqrt[3]{1-\lambda^2}, \sqrt[3]{-1+2\lambda^2+\sqrt{1-\lambda^2}}\right\}. \tag{4.52}$$

Proof. Let $0 < \lambda < 1$, and consider the real polynomials

$$\begin{aligned}
g_2(x) = \quad & x^2 - 2\sqrt{1-\lambda^2}x + 1 = (x - \sqrt{1-\lambda^2} - i\lambda)(x - \sqrt{1-\lambda^2} + i\lambda)\ , \\
g_3(x) = \quad & g_2(x)(x - \sqrt{1-\lambda^2}) = x^3 - x^2 3\sqrt{1-\lambda^2} + x(3-2\lambda^2) - \sqrt{1-\lambda^2}, \\
g_4(x) = \quad & g_2(x)(x^2 + \sqrt{1-\lambda^2}x + 1) = x^4 - x^3\sqrt{1-\lambda^2} + x^2 2\lambda^2 - z\sqrt{1-\lambda^2} + 1\ , \\
g_5(x) = \quad & g_2(x)(x^3 + \sqrt{1-\lambda^2}x^2 + \sqrt{1-\lambda^2}x + 1) \\
& = x^5 - x^4\sqrt{1-\lambda^2} + (x^3 + x^2)(-1 + 2\lambda^2 + \sqrt{1-\lambda^2}) - x\sqrt{1-\lambda^2} + 1\ ,
\end{aligned} \tag{4.53}$$

which were chosen so each have the same Turán ratios. The lone Turán ratio of $g_2(x)$ is $\beta_2 = 4(1-\lambda^2)$. The sequence of Turán ratios of $g_3(x)$ is

$$\left\{\frac{(3-2\lambda^2)^2}{3(1-\lambda^2)}, \frac{9(1-\lambda^2)}{3-2\lambda^2}\right\}. \tag{4.54}$$

Setting the expressions in (4.54) equal to one another yields the lone Turán constant $\beta_3 = 3\sqrt[3]{1-\lambda^2}$.

The sequence of Turán ratios of $g_4(x)$ is

$$\left\{ \frac{1-\lambda^2}{2\lambda^2}, \frac{4\lambda^4}{1-\lambda^2}, \frac{1-\lambda^2}{2\lambda^2} \right\}. \tag{4.55}$$

Setting the expressions in (4.55) equal to one another yields the lone Turán ratio $\beta_4 = \sqrt{2}\lambda$. The sequence of Turán ratios of $g_4(x)$ is

$$\left\{ \frac{1-\lambda^2}{-1+2\lambda^2+\sqrt{1-\lambda^2}}, -\frac{-1+2\lambda^2+\sqrt{1-\lambda^2}}{\sqrt{1-\lambda^2}}, -\frac{-1+2\lambda^2+\sqrt{1-\lambda^2}}{\sqrt{1-\lambda^2}}, \frac{1-\lambda^2}{-1+2\lambda^2+\sqrt{1-\lambda^2}} \right\},$$

and setting these expressions equal to one another yields the lone Turán ratio $\beta_5 = \sqrt[3]{-1+2\lambda^2+\sqrt{1-\lambda^2}}$.

Next, we show that the Turán ratios we have constructed above are indeed the lower bounds on the Turán ratios of real and highly-concave polynomials, all of whose zeros lie in $S(\lambda)$, of degree at least 5. To this end, let $N-k, k = 2, 3, 4, 5$, be a fixed positive integer greater than or equal to 5. By Lemma 102, as $n \to \infty$, the minimal Turán constant of any polynomial $f_N(x) = g_k(x) \cdot p_{n,N-k}(x)$, where for some $n \in \mathbb{R} \smallsetminus \{0\}$ the Turán ratios of $p_{n,N-k}(x)$ are all equal to $\frac{1}{n^2}$, is at least as large as the Turán constant of $g_k(x)$. Thus, if the zeros of any such $f_N(x)$ all lie in $S(\lambda)$, we have the bound $\beta_{f_N,S(\lambda)}$ given in (4.52). □

Corollary 104. *Let U be an open connected set of polynomials and let $S(\lambda), 0 < \lambda < 1$, be the open strip $\{z \in \mathbb{C} \mid |\Im z| < \lambda\}$. If U contains one of the polynomials listed in* (4.53)*, then all zeros of all elements of U lie in $S(\lambda)$.*

Proof. We observe that each of the polynomials in Lemma 103 have zeros $\sqrt{1-\lambda^2} \pm i\lambda$, where $0 < \lambda < 1$, and no zeros outside of $S(\lambda), 0 < \lambda \leq 1$. Indeed, the non-real zeros of $x^2 + \sqrt{1-\lambda^2}x + 1$ and $x^3 + (x^2+x)\sqrt{1-\lambda^2} + 1$, are given by

$$\frac{-\sqrt{1-\lambda^2} \pm \sqrt{-3-\lambda^2}}{2}, \ \frac{1-\sqrt{1-\lambda^2} \pm \sqrt{-4+\left(-1+\sqrt{1-\lambda^2}\,\right)^2}}{2}, \tag{4.56}$$

respectively, and, for $0 < \lambda < 1$, lie in $S(\lambda)$. Hence, by Lemma 100, all elements of U have all their zeros in $S(\lambda)$. □

Corollary 105. *Let $\varphi(x) = \sum_{k=0}^{N} a_k x^k + \sum_{k=N+1}^{\omega} a_k x^k, 2 \le \omega \le \infty$, be a real entire function. Suppose that $\varphi(x)$ generates a highly-concave sequence of positive coefficients whose associated sequence of Turán ratios is constant. If the zeros of $\varphi(x)$ and some partial sum $\sum_{k=0}^{N} a_k x^k, 2 \le N \le 5$, all lie in the open strip $S(\lambda), 0 < \lambda < 1$, then $\beta_{\varphi,S(\lambda)}$ is constrained by (4.52).* □

4.1.6 Iterating

In Section 3.2.2 we have encountered the Toeplitz-type non-linear operators and their iterations. We investigate here the stability preserving properties of these operators and their iterates.

Proposition 106. *Let $\{a_k\}_{k=0}^{\infty}$ be a highly log-concave sequence of positive numbers, such that for some $\beta > 1$ and $k = 3, 4, 5, \ldots$, the inequalities $a_k^2 \le \beta^4 a_{k-2} a_{k+2}$ hold. Then the sequence $\{\frac{a_k^2}{a_{k-1}a_{k+1}}\}_{k=1}^{\infty}$ is log-concave.*

Proof. Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of positive numbers and suppose that $a_k^2 \ge \beta a_{k-1} a_{k+1}$, for some $\beta > 1$ and all positive integers k. Then,

$$\frac{a_{k-1}^2}{a_{k-2}a_k} \cdot \frac{a_{k+1}^2}{a_k a_{k+2}} \ge \beta^2, \tag{4.57}$$

and whence, we need to show that, for each positive integer k, we have the inequalities

$$\frac{a_k^4}{a_{k-1}^2 a_{k+1}^2} - \frac{a_{k-1}^2}{a_{k-2}a_k} \cdot \frac{a_{k+1}^2}{a_k a_{k+2}} \ge \beta^2 - \frac{a_{k-1}^2}{a_{k-2}a_k} \cdot \frac{a_{k+1}^2}{a_k a_{k+2}} \ge 0\ . \tag{4.58}$$

Next, for each positive integer k, from the hypothesis that $\{a_k\}_{k=0}^{\infty}$ is highly log-concave, we obtain the inequalities

$$\begin{aligned} a_k^4 &\ge \beta^2 a_{k-1}^2 a_{k+1}^2 \\ &\ge \beta^2 (\beta a_{k-2} a_k)(\beta a_k a_{k+2}) \\ &= \beta^4 a_k^2 a_{k-2} a_{k+2}, \end{aligned} \tag{4.59}$$

but by hypothesis this implies that $a_k^2 = \beta^4 a_{k-2} a_{k-2}$. Now, we estimate the right member of (4.58),

$$\begin{aligned} \beta^2 - \frac{a_{k-1}^2}{a_{k-2}a_k} \cdot \frac{a_{k+1}^2}{a_k a_{k+2}} &= \beta^2 - \frac{\beta^4 a_{k-1}^2 a_{k+1}^2}{a_k^4} \\ &\ge \beta^2 - \beta^2. \end{aligned} \tag{4.60}$$

Thus, from (4.58) we conclude that, for each positive integer k,

$$\frac{a_k^4}{a_{k-1}^2 a_{k+1}^2} - \frac{a_{k-1}^2}{a_{k-2}a_k} \cdot \frac{a_{k+1}^2}{a_k a_{k+2}} \geq 0 \ . \tag{4.61}$$

$\square$

Remark 107. We note that the hypothesis in the above proposition may be weakened to $\beta = 1$.

Lemma 108. *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of positive numbers such that $\frac{a_k^2}{a_{k-1}a_{k+1}} \leq 1$ for all positive integers k. Then the sequence $\{1 - \frac{1}{a_k}\}_{k=0}^{\infty}$ is $\sqrt{x_0}$-log-concave provided that $a_k \leq y_0 \approx 0.221377$, the positive zero of $x^2(1-\sqrt{x_0}) - 2x + 1 - \sqrt{x_0}$, where x_0 is the positive zero of $x^3 - x^2 - 2x - 1$.*

Proof. A direct calculation for $k = 1, 2, 3, \ldots$, yields

$$\begin{aligned}
\frac{\left(1 - \frac{1}{a_k}\right)^2}{\left(1 - \frac{1}{a_{k-1}}\right)\left(1 - \frac{1}{a_{k+1}}\right)} &= \frac{a_k^2 - 2a_k + 1}{\frac{a_k^2}{a_{k-1}a_{k+1}} \cdot (1 - a_{k-1} - a_{k+1} + a_{k-1}a_{k+1})} \\
&\geq \frac{a_k^2 - 2a_k + 1}{\frac{a_k^2}{a_{k-1}a_{k+1}} \cdot (1 + a_{k-1}a_{k+1})} \\
&\geq \frac{a_k^2 - 2a_k + 1}{a_k^2 + 1}.
\end{aligned} \tag{4.62}$$

Now, $\frac{a_k^2 - 2a_k + 1}{1 + a_k^2} \geq \sqrt{x_0}$ provided $a_k^2(1 - \sqrt{x_0}) - 2a_k + 1 - \sqrt{x_0} \geq 0$ for all positive integers k. The real roots of this quadratic polynomial in the indeterminate a_k lie at about $a_k = 0.221377$ and $a_k = -4.51718$. $\square$

Lemma 109. *Let $\varphi(x)$ be a (Hurwitz) stable real polynomial that generates a sequence of Turán ratios bounded by the positive root of $x^2(1-\sqrt{x_0}) - 2x + 1 - \sqrt{x_0}$, where x_0 is the positive root of $x^3 - x^2 - 2x - 1$, and whose sequence of Turán ratios is bounded by 1. Then the Hadamard product of $T^3[\varphi(x)]$ and $\Lambda^1(z^2+1)[\varphi(x)]$ is (Hurwitz) stable.*

Proof. Let $\varphi(x) = \sum_{k=0}^{\omega} a_k x^k, 0 \leq \omega < \infty$, be a real stable polynomial. Suppose that $\{\alpha_k := \frac{a_k^2}{a_{k-1}a_{k+1}}\}_{k=1}^{\infty}$ is a sequence bounded by r_0, the positive root of $x^2(1-\sqrt{x_0}) - 2x + 1 - \sqrt{x_0}$, where x_0 is the positive root of $x^3 - x^2 - 2x - 1$, and that the associated sequence of Turán constants $\{\frac{\alpha_k^2}{\alpha_{k-1}\alpha_{k+1}}\}_{k=2}^{\infty}$ is bounded by 1. By Lemma 108, the sequence $\{1 - \frac{1}{\alpha_k}\}_{k=1}^{\omega-1}$ is $\sqrt{x_0}$–log-concave, and whence by Theorem 78, the function $\sum_{k=0}^{\omega-1}\left(1 - \frac{1}{\alpha_k}\right) x^k$ is stable. Now, because φ is stable, so is $\Lambda^1(z^2+1)[\Lambda^1(z^2+1)[\varphi(x)]]$ and, too, the Hadamard product (cf. §1.6.2),

$$\begin{aligned}\sum_{k=0}^{\omega} a_k x^k \star \sum_{k=0}^{\omega} T_k^2[T_k^2[a_k]]x^k \star \sum_{k=0}^{\omega}\left(1-\frac{1}{\alpha_k}\right)x^k &= \sum_{k=0}^{\omega} a_k \cdot a_k T_k^3[a_k] \cdot \frac{a_k^2 - a_{k-1}a_{k+1}}{a_k^2} x^k \\ &= \sum_{k=0}^{\omega} T_k^3[a_k] \cdot \Lambda_k^1(z^2+1)[a_k]x^k \\ &= T^3[\varphi(x)] \star \Lambda^1(z^2+1)[\varphi(x)].\end{aligned} \tag{4.63}$$

□

4.1.7 Infinite log-concavity

It is now time to consider the concavity properties of the sequence $\{a_k^2 - a_{k-1}a_{k+1}\}_{k=0}^{\infty}$, which may be viewed as the first iterate of the sequence $\{a_k\}_{k=0}^{\infty}$ under the non-linear operator $a_k \mapsto a_k^2 - a_{k-1}a_{k+1}$. The study of log-concavity under various non-linear operators and iterations requires its own terminology.

Definition 110. A sequence of positive numbers whose n^{th}-iterate under a non-linear operator $a_k \mapsto a_k^2 - a_{k-1}a_{k+1}$ is (highly) log-concave will be called a *n-fold* (highly) log-concave sequence. □

Brändén's Theorem (Theorem 35) provides a class of examples of *infinitely log-concave* sequences, the principal example of which are the binomial coefficients. Indeed, the non-linear operator $\Lambda^1(z^2+1)$ takes elements $\mathscr{L}$-$\mathscr{P}^+$ back into $\mathscr{L}$-$\mathscr{P}^+$, so in particular the coefficients of $(x+1)^n$, for each positive integer n, are log-concave after each application of $\Lambda^1(z^2+1)$. We give here another example of infinite log-concavity, which appears to be new.

Proposition 111. *The Stirling numbers of the first and second kind are infinitely log-concave.*

Proof. Let $B_n(x)$ be the polynomial of degree n generated by $e^x B_n(x) = \sum_{k=0}^{\infty} \frac{k^n}{k!} x^k$. The polynomials $B_n(x)$ are known as the Bell polynomials and their coefficients are given by the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$, which represent the number of ways a set of n elements can be partitioned into $k, 0 \le k \le n$, non-empty subsets (see [8] for an excellent review of the results used here). It is known that the zeros of the Bell polynomials are all real, simple, and non-positive, and form Brändén's Theorem and the fact that for each positive integer n,

$$B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k, \tag{4.64}$$

it follows that for each positive integer n, the sequence $\left\{\left\{{n \atop k}\right\}\right\}_{k=0}^{n}$ is infinitely log-concave.

Similarly, the (unsigned) Stirling numbers of the first kind, denoted $\left[{n \atop k}\right]$, are infinitely log-concave. These are the coefficients of the polynomials

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{4.65}$$

□

G. Boros and V. Moll ([7]), who have studied the polynomials $P_m(a), a > -1$, given by

$$P_m(a) = \frac{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}}{\pi} \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \tag{4.66}$$

and their coefficients $d_k(m)$, where

$$\begin{aligned} d_k(m) &= \frac{1}{4^m} \sum_{j=k}^{m} 2^j \binom{2m-2j}{m-j} \binom{m+j}{m} \binom{j}{k} \\ &= \frac{1}{4^m} \sum_{j=0}^{m-k} 2^{j-k} \binom{2m-2k-2j}{m-k-j} \binom{m+k+j}{j+k} \binom{j+k}{k}, \end{aligned} \tag{4.67}$$

conjectured that the sequence $\{d_k(m)\}_{k=0}^{m}$ is infinitely log-concave for all positive integers m. P. Brändén conjectured in [9] that $\sum_{k=0}^{m} \frac{d_k(m)}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P}^+$, for each positive integer m, and that $\sum_{k=0}^{m} \frac{d_k(m)}{(k+2)!} x^k \in \mathscr{L}\text{-}\mathscr{P}^+$, which would imply, by [21, Theorem 5.5], the 2-fold and 3-fold log-concavity of $\{d_k(m)\}_{k=0}^{\infty}$. Now, it is known that $\{d_k(m)\}_{k=0}^{\infty}$ is log-concave and 2-fold log-concave (see [15], [60]), although not by means of P. Brändén's conjecture. Nevertheless, the Boros-Moll conjecture, which states that , remains unresolved.

A formula relating the Stirling numbers of the first and second kinds (see [1, §21.1.4]) bears some resemblance to the Boros-Moll sequences (4.67),

$$\left\{ {m \atop k} \right\} = \sum_{j=0}^{m-k} (-1)^{m-k+j} \binom{m-1-j}{m-k+j} \binom{2m-k}{m-k-j} \begin{bmatrix} m-k+j \\ j \end{bmatrix}, \tag{4.68}$$

and if there is more than just a superficial connection here, then it will be the combinatorial, rather than the analytical approach that will reveal it.

Recently, D. Karp ([57]) showed that if $\{f_k\}_{k=0}^{n}$ is a log-concave sequence of real numbers, then the coefficients of the polynomial,

$$\sum_{k=0}^{n} f_k f_{n-k} \binom{n}{k} \left[(x)_k (x)_{n-k} - (x+1)_k (x-1)_{n-k}\right], \tag{4.69}$$

are all positive for each positive integer n and conjectured that for $n \geq 3$ the zeros of the above polynomials are in fact all real and negative, provided $\{f_k\}_{k=0}^{n}$ is a totally positive sequence. We eagerly await the resolution of this conjecture, because of the representation of the coefficients of (4.69) in terms of the Stirling numbers.

Computer experiments suggest that the zeros of the polynomials $\sum_{\ell=0}^{n-2} P_{j,m,n} x^{\ell}$, where $P_{j,m,n}$ is given by

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{k} \sum_{j=0}^{m} \begin{bmatrix} k-1 \\ j \end{bmatrix} \left((n(n-1) - 2k(n-k)) \begin{bmatrix} n-k-1 \\ m-j+1 \end{bmatrix} - (n(n-1) - 4k(n-k)) \begin{bmatrix} n-k-j \\ m-j \end{bmatrix} \right), \tag{4.70}$$

all are real and negative for each choice of integers $0 \leq m \leq n-2$. The numbers $P_{j,m,n}$ appear in the coefficient-wise-transformation representation of

$$\sum_{k=0}^{n} f_k f_{n-k} \binom{2n}{k} \left[(x)_k (x)_{n-k} - (x+1)_k (x-1)_{n-k}\right], \tag{4.71}$$

which is given by

$$\{f_k\}_{k=0}^{n} \mapsto \left\{ \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor - 1} f_j f_{n-j} P_{j,m,n} \right\}_{m=0}^{n-2}. \tag{4.72}$$

The conjecture of D. Karp is known to fail if $\{f_k\}_{k=0}^{n}$ is not totally positive, but if we consider the polynomials in (4.70) and ignore the coefficients of the terms $f_0 f_n$ and $f_1 f_{n-1}$ as in (4.72), then the numerical experiments mentioned above suggest that their zeros are all real an negative provided only that $\{f_k\}_{k=0}^{n}$ is log-concave.

4.2 Zeros of sums of hypergeometric polynomials

We recall that in 1946 P. Turán obtained the following inequality for Legendre polynomials (see for example [82]),

$$\left({}_2F_1(-\mu, 1+\mu; 1; x)\right)^2 \geq {}_2F_1(-\mu-\delta, 1+\mu+\delta; 1; x)\,{}_2F_1(-\mu+\delta, 1+\mu-\delta; 1; x), \tag{4.73}$$

where $x \in (0,1), \mu = 1,2,3,\ldots,$ and $\delta = 1$. D. Karp conjectured that (4.73) holds for all $\mu > 0$ and $0 < \delta < 1$ ([56]). The bounds on the elements of the sequence that generates non-linear operators $\Lambda^1(z^2+1)$ can be used to study the distribution of zeros of certain sums of hypergeometric polynomials.

Proposition 112. *Let $\mu_0, \mu_2 \in \mathbb{C}$. Then, for any positive integer n,*

$$\begin{aligned}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k (k(\mu_0-\mu_2)+\mu_0) z^k = \frac{z}{2}(\mu_0-\mu_2)n(n-1)\ {}_2F_1\left(\frac{3}{2}-\frac{n}{2}, 1-\frac{n}{2}; 3; 4z\right) \\ + \mu_0\ {}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; 2; 4z\right)\ ,\end{aligned} \tag{4.74}$$

where $C_k = \frac{1}{1+k}\binom{2k}{k}, k = 0,1,2,\ldots,$ are the Catalan numbers.

Proof. Fix complex numbers μ_0, μ_2, and a positive integer n.

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k (k(\mu_0-\mu_2)+\mu_0) z^k = (\mu_0-\mu_2) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} k \binom{n}{2k} C_k z^k + \mu_0 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k z^k\ , \tag{4.75}$$

and applying Lemma 44, yields

$$\mu_0 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k z^k = \mu_0\ {}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; 2; 4z\right)\ . \tag{4.76}$$

Then, using (2.13)

$$\begin{aligned}
\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} k \binom{n}{2k} C_k z^k &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k} \frac{k}{k+1} z^k \\
&= \sum_{k=-1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k+2}\binom{2k+2}{k+1} \frac{k+1}{k+2} z^{k+1} \\
&= z \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!(n-2k)!} \cdot \frac{(n-2k)(n-2k-1)}{(k+1)!(k+2)} z^k \\
&= z \sum_{k=0}^{\infty} \frac{(-n)_{2k}}{k!(2)_k} \cdot \frac{(n-2k)(n-2k-1)}{(k+2)} z^k \\
&= z \sum_{k=0}^{\infty} \frac{\left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k 2^{2k}}{k!(2)_k} \cdot \frac{(n-2k)(n-2k-1)}{(k+2)} z^k \\
&= z \sum_{k=0}^{\infty} \frac{\left(\frac{3-n}{2}\right)_k \left(1-\frac{n}{2}\right)_k}{k!(3)_k} \frac{n(n-1)}{2} (4z)^k \\
&= \frac{z}{2} n(n-1)\ {}_2F_1\left(\frac{3}{2}-\frac{n}{2}, 1-\frac{n}{2}; 3; 4z\right) .
\end{aligned} \tag{4.77}$$

□

4.3 Open problems, conjectures, and further research

This section is dedicated to questions, at which we wonder excessively, and problems which we attack without mercy or provocation, yet which remain unresolved.

The examples of non-linear coefficient-wise stability and hyperbolicity preservers we have considered in the prequel by no means constitute a complete understanding of such objects.

Problem 113. *Characterize the non-linear coefficient-wise transformations that map real polynomials with only real zeros into polynomials of the same type.* □

In Section 4.1.3 we considered the action of the non-linear operator $S_r : a_k \mapsto a_k^2 - a_{k-r}a_{k+r}$ and exhibited in Example 85 a function not belonging to $\mathscr{L}\text{-}\mathscr{P}^+$ that is mapped by S_r into $\mathscr{L}\text{-}\mathscr{P}^+$. We are interested in the *rigidity* of the Laguerre-Pólya class.

Problem 114. *Let Λ be a non-linear operator acting on the coefficients of a real entire function and suppose that Λ preserves a subclass of (or all of) $\mathscr{L}\text{-}\mathscr{P}^+$. Classify the real entire functions $f(x) \notin \mathscr{L}\text{-}\mathscr{P}$ for which $\Lambda[f(x)] \in \mathscr{L}\text{-}\mathscr{P}^+$.* □

As a broad generalization of Theorem 42 and Problem 65, we make the following conjecture.

Conjecture 115. *Let $g(z)$ be the generating polynomial of the system of inequalities (1.7) that characterize $\mathscr{L}$-$\mathscr{P}$. The class of non-linear operators $\Lambda_k^p(g)$ preserves $\mathscr{L}$-$\mathscr{P}^+$.* $\square$

The following Theorem used to be known as the Hawaii Conjecture.

Theorem 116 ([87]). *Let $\varphi \in \mathscr{L}$-$\mathscr{P}^*$ and suppose that φ has $2m$ non-real zeros. Then $\left(\frac{\varphi'}{\varphi}\right)'$ has at most $2m$ real zeros.* $\square$

For hyperbolic polynomials p, the expression $pp'' - \frac{n-1}{n}(p')^2$ may be regarded as a sharpened form of the Turán expression and is related to the Newton inequalities that follows from the ultra log-concavity of the associated sequence of coefficients. The B. Shapiro conjectured that the following sharpened version of the Theorem 116 holds.

Conjecture 117 (B. Shapiro). *Let $p \in \mathbb{R}[x]$ be a polynomial of degree n with $2m$ non-real zeros. Then,*

$$\frac{pp'' - \frac{n-1}{n}(p')^2}{p^2}, \tag{4.78}$$

has at most $2m$ real zeros. $\square$

If the Shapiro Conjecture is true, it does not imply Theorem 116. Note also that according to Corollary 55, the non-linear operator $a_k \mapsto \frac{n-1}{n}a_k^2 + a_{k-1}a_{k+1}$, obtained from the numerator of (4.78) evaluated at $x = 0$, is a hyperbolicity preserver.

BIBLIOGRAPHY

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions,* Dover, New York, 1965.

[2] M. Aissen, I. J. Schoenberg and A. M. Whitney, *On generating functions of totally positive sequences I,* J. Anal. Math. **2** (1952), 93-103.

[3] R. P. Boas, Jr., *George Pólya: Collected Papers, vol. 2,* MIT Press, Cambridge, Mass., 1974.

[4] J. Borcea, A. Guterman, and B. Shapiro, *Preserving positive polynomials and beyond,* arXiv:0801.1749v2[math.CA], 2008.

[5] J. Borcea and P. Brändén, *Pólya-Schur master theorems for circular domains and their boundaries,* Ann. of Math. **170** (2009), 465-492.

[6] J. Borcea, P. Brändén, G. Csordas, and V. Vinnikov, *Pólya-Schur-Lax problems: hyperbolicity and stability preservers,* AIM workshop, 2007.

[7] G. Boros and V. Moll, *Irresitible integrals,* Cambridge University Press, Cambridge, 2004.

[8] K. N. Boyadzhiev, *Exonential polynomials, Stirling numbers, and evaluation of some gamma integrals,* Abstract and Applied Analysis **2009**, Article ID 168672, 2009.

[9] P. Brändén, *Iterated sequences and the geometry of zeros,* J. Reine Angew. Math., (to appear), arXiv:0909.1927v3[math.CO], 2010.

[10] F. Brenti, *Unimodal, log-concave and Pólya frequency sequences in combinatorics,* Mem. Amer. Math. Soc. **81** (1989), viii-106.

[11] D. A. Cardon, *Extended Laguerre inequalities and a criterion for real zeros,* arXiv:0911.1122v1[math.CV], 2009.

[12] D. A. Cardon and A. Rich, *Turán inequalities and subtraction-free expressions,* J. Ineqal. Pure Appl. Math. **9** (2008), Article 91.

[13] L. Carlitz, *Note on a formula of Szily,* Scripta Math. **18** (1953), 249-253.

[14] M. Chasse, *Linear preservers and entire functions with restricted zero loci,* Ph. D. dissertation, University of Hawai'i, Honolulu, HI, 2011.

[15] W. Y. C. Chen and E. X. W Xia, *2-Log-concavity of the Boros-Moll Polynomials,* arXiv:1010.0416v1[math.CO], 2010.

[16] J. B. Conway, *Functions of One Complex Variable,* Springer-Verlag, New-York, second edition, 1978.

[17] T. Craven and G. Csordas, *The Gauss-Lucas theorem and Jensen polynomials*, Trans. Amer. Math. Soc. **278** (1983), 415-429.

[18] T. Craven and G. Csordas, *Jensen Polynomials and the Turán and Laguerre inequalities,* Pacific J. Math. **136** (1989), 241-260.

[19] T. Craven and G. Csordas, *Complex zero decreasing sequences*, Methods Appl. Anal. **2** (1995), 420-441.

[20] T. Craven and G. Csordas, *Problems and theorems in the theory of multiplier sequences,* Serdica Math. J. **22** (1996), 515-524.

[21] T. Craven and G. Csordas, *Iterated Laguerre and Turán inequalities,* Journal of Inequalities in Pure and Applied Mathematics. **3**, Issue 3 (2002), Article 39.

[22] T. Craven and G. Csordas, *Composition theorems, multiplier sequences and complex zero decreasing sequences,* in Value Distribution Theory and Related Topics, (G. Barsegian, I. Laine and C. C. Yang, eds.), Adv. Complex. Anal. Appl. **3**, 131-166, Kluwer Acad. Publ., Boston, MA, 2004.

[23] G. Csordas, *Necessary and sufficient conditions and the Riemann hypothesis,* Adv. in Appl. Math. **11** (1990), 328-357.

[24] G. Csordas, *Convexity and the Riemann ξ-function,* Glasnik Matematički **33** (1998), 37-50.

[25] G. Csordas, *Complex zero decreasing sequences and the Riemann hypothesis II,* in Analysis and its Applications (Begehr, Gilbert, Wong, eds.), Kluwer Academic Publishers (2001), 121-134.

[26] G. Csordas, *Linear operators, Fourier transforms and the Riemann ξ-function,* in Some Topics in Value Distribution and Differentiability in Complex and P-adic Analysis, 188-207, Science Press, Beijing, 2008.

[27] G. Csordas and M. Chasse, *Discrete analogues of the Laguerre inequalities and a conjecture of I. Krasikov,* Ann. Sci. Math. Quèbec, (to appear).

[28] G. Csordas and A. Escassut, *The Laguerre inequality and the distribution of zeros of entire functions.* Ann. Math. B. Pascal, **12** (2005), 331-345.

[29] G. Csordas, T. S. Norfolk, and R. S. Varga, *The Riemann hypothesis and the Turán inequalities,* Trans. Amer. Math. Soc. **296** (1986), 521-541.

[30] G. Csordas and R. S. Varga, *Necessary and sufficient conditions and the Riemann hypothesis,* Adv. Appl. Math. 11 (1990), 328-357.

[31] H. Davenport and G. Pólya, *On the product of two power series,* Canad. J. Math. **1** (1949), 1-5.

[32] L. S. Dederick, *Successive Derivatives of Functions of Several Functions,* Ann. of Math., **17** (1926), 385-394.

[33] D. K. Dimitrov, *Higher Order Turán Inequalities,* Proc. Amer. Math. Soc. **126** (1998), 2033-2037.

[34] K. Driver and K. Jordaan, *Zeros of the hypergeometric polynomial F(-n,b; c; z),* arXiv:0801.0708v1 [math CA], 2008.

[35] A. Edrei, *Proof of a conjecture of Schoenberg on the generating function of a totally positive sequence,* Canad. J. Math. **5** (1953), 86-94.

[36] S. Fisk, *Questions about determinants and polynomials,* arXiv:0808.1850v1[math.CA], 2008.

[37] S. Fisk, *Polynomials, roots, and interlacing,* arXiv:math/0612833v2 [math.CA], 2008.

[38] W. H. Foster and I. Krasikov, *Inequalities for real-root polynomials and entire functions,* Adv. in Appl. Math. **29** (2002), 102-114.

[39] L. Gårding, *An inequality for hyperbolic polynomials,* J. Math. Mech. **8** (1959), 957-965.

[40] F. R. Gantmacher, *The Theory of Matrices,* Vol. 2, Chelsa, New York, 1959.

[41] J. Garloff and D. G. Wagner, *Hadamard products of stable polynomials are stable,* J. Math. Anal. Appl. **202** (1996), 797-809.

[42] J. C. F. Gauss, *Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse,* Ph. D. dissertation, Universitat Helmstedt, 1799.

[43] L. Grabarek, *A new class of non-linear stability preserving operators,* Complex Variable and Elliptic Equations: An International Journal, iFirst, DOI:10.1080/17476933.2011.586696 (2011), 1-12.

[44] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products,* Academic Press, San Diego, 1980.

[45] J. Hadamard, *Theéorème sur les séries entières,* Acta. Math. 22 (1899), 55-64.

[46] D. Handelman, *Arguments of zeros of highly log concave polynomials,* arXiv:1009.6022v1[math.CA], 2010.

[47] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities,* Combridge University Press, Cambridge, 1952.

[48] O. Holtz, *Hermite-Biehler, Routh-Hurwitz, and total positivity,* Linear Algebra Appl. **372** (2003), 105-110.

[49] R. M. Hovstad, *A short proof of a continued fraction test for the stability of polynomials,* Proc. Amer. Math. Soc. **105** (1989), 76-79.

[50] A. Hurwitz, *Über die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt,* Mathematische Annalen, **46** (2), 273-284.

[51] J. I. Hutchinson, *On a remarkable class on entire functions,* Trans. Amer. Math. Soc. **25** (1923), 325-322.

[52] N. Jacobson, *Lectures in Abstract Algebra,* Vol. III, D. Vab Bostrand Company, Inc., Princeton, NJ, 1964.

[53] J. L. W. V. Jensen, *Reserches sur la théorie des équations,* Acta. Math. 36 (1912-1913), 181-195.

[54] W. P. Johnson, *The Curious History of Faà di Bruno's Formula,* Amer. Math. Monthly **109** (2002), 217-234.

[55] S. Karlin, *Total Positivity, Vol. 1,* Stanford University Press, Stanford, California, 1968.

[56] D. Karp, *Log-convexity and log-concavity for series in product ratios of rising factorials and gamma functions,* Special Functions in the 21st Century: Theory and Applications, 6-8, 2011, Washington DC, DC, USA. Available at http://math.nist.gov/ DLozier/SF21/.

[57] D. Karp, *Positivity of Toeplitz determinants formed by rising factorial series and properties of related polynomials,* arXiv:1203.1482v1[math.CA], 2012.

[58] O. Katkova, *Multiple positivity and the Riemann zeta-function,* Comput. Methods Funct. Theory **7** (2007), 13-31.

[59] O. M. Katkova and A. M. Vishnyakova, *A sufficient condition for a polynomial to be stable,* J. Math. Anal. Appl. **349** (2007), 81-89.

[60] M. Kauers and P. A. Paule, *A computer proof of Moll's log-concavity conjecture,* Proc. Amer. Math. Soc. **135** (2007), 3847-3856.

[61] V. P. Kostov and B. Shapiro, *Hardy-Petrovitch-Hutchinson's problem and partial theta functions,* arXiv:1106.6262v1[math.CA], 2011.

[62] I. Krasikov, *Discrete analogues of the Laguerre inequality,* Anal. Appl. (Singap.) **1** (2003), 189-197. 92

[63] D. C. Kurtz, *A sufficient condition for all the roots of a polynomial to be real,* Amer. Math. Monthly **99** (1992), 259-263.

[64] E. Laguerre, *Sur quelques points de la théorie des équations numeriques,* Acta. Math. **4** (1884), 97-120.

[65] B. Ja. Levin, *Distribution of Zeros of Entire Functions,* Transl. Math. Mono. **5**, Amer. Math. Soc., Providence, RI, 1964; Revised Ed., 1980.

[66] T. Ligget, *Ultra logconcave sequences and negative dependence,* J. Combin. Theory Ser. A **79 (2)** (1997), 315-325.

[67] M. Marden, *Geometry of Polynomials,* Math. Surveys no. 3, Amer. Math. Soc., Providence, RI, 1966.

[68] A. I. Markushevich, *Theory of Functions of a Complex Variable,* Vol. I-III, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1965.

[69] P. R. W. McNamara and B. E. Sagan, *Infinite log-concavity: developments and conjectures,* arXiv:0808.1065v2[math.CO], 2009.

[70] P. Montel, *Leçons sur les familles normales de foncions analytiques*, Paris 1927.

[71] N. Obreschkoff, *Verteilung und Berechnung der Nullstellen reeller Polynome,* VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[72] M. Passarw, J. M. Rojas, and B. Shapiro, *New multiplier sequences via discriminant amoebae,* arXiv:1001.4076v1 [math.AG], 2010.

[73] M. L. Patrick, *Extensions of inequalities of the Laguerre and Turán type,* Pacific J. Math. **44** (1973), 675-682.

[74] G. Pólya, *Collected Papers, Vol. II Location of Zeros,* (R. P. Boas, ed.), MIT Press, Cambridge, MA, 1974.

[75] G. Pólya and J. Schur, *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen,* J. Reine Angew. Math., **144** (1914), 89-113.

[76] G. Pólya and G. Szegö, *Problems and Theorems in Analysis, Vols. I and II,* Springer-Verlag, New York, 1976.

[77] A. Pinkus, *Some remarks on zero-increasing transformations,* in Approximation Theory, X (St. Louis, MO, 2001), Innov. Appl. Math., 333-352, Vanderbilt Univ. Press, Nashville, TN, 2002.

[78] A. Pinkus, Totally Positive Matrices, Cambridge University Press, Cambridge, 2010.

[79] A. Piotrowski, *Linear operators and the distribution of zeros of entire functions,* Ph.D. dissertation, Univeristy of Hawai'i, Honolulu, HI, 2007.

[80] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials,* Oxford University Press, 2002.

[81] E. Rainville, *Special Functions,* Chelsea, New York, 1960.

[82] H. Skovgaard, *On inequalities of Turán type,* Math. Scand. **2** (1954), 65-73.

vol. 576 of Ann. New York Acad. Sci., New York Acad. Sci., New York (1989), 500-535.

[83] Sh. Strelitz, *On the Routh-Hurwitz Problem,* Amer. Math. Monthly, **84** (1977), 542-544.

[84] O. Szász, *On sequences of polynomials and the distribution of their zeros,* Bull. Amer. Math. Soc. **49** (1943), 377-383.

[85] K. v. Szily, *Über die Quadratsummen der Binomialcoefficienten,* Mat. Nat. Ber. Ungarn. **12** (1894), 84-91.

[86] E. C. Titchmarsh, *The Theory of Functions,* Oxford University Press, Second Ed., 1939. Reprint, 1964.

[87] M. Tyaglov, *On the number of real critical points of logarithmic derivatives and the Hawaii conjecture,* J. Anal. Math. **114** (2011), 1-62.

[88] D. G. Wagner, *Total positivity and Hadamard products,* J. Math. Anal. Appl. **163** (1992), 459-483.

[89] J. N. Whitley, *On Newton's inequality for real polynomials,* Amer. Math. Monthly **76** (1969), 905-909.

[90] D. V. Widder, *The Laplace Transform,* Princeton Univ. Press, Princeton, 1941.

[91] R. Yoshida, *On some questions of Fisk and Brändén,* Complex Variable and Elliptic Equations: An International Journal, iFirst, DOI:10.1080/17476933.2011.603418 (2011), 1-13.

[92] R. Yoshida, Personal communication, 2011.